Punctured holomorphic curves and Lagrangian embeddings

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Abstract We use a neck stretching argument for holomorphic curves to produce symplectic disks of small area and Maslov class with boundary on Lagrangian submanifolds of nonpositive curvature. Applications include the proof of Audin's conjecture on the Maslov class of Lagrangian tori in linear symplectic space, the construction of a new symplectic capacity, obstructions to Lagrangian embeddings into uniruled symplectic manifolds, a quantitative version of Arnold's chord conjecture, and estimates on the size of Weinstein neighbourhoods. The main technical ingredient is transversality for the relevant moduli spaces of punctured holomorphic curves with tangency conditions.

1 Introduction

In this paper we use punctured holomorphic curves to establish some new restrictions on Lagrangian embeddings. We will denote by *manifold of nonnegative curvature* a manifold which admits a Riemannian metric of nonpositive

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sectional curvature. In fact, the only property that enters the proofs is the existence of a metric for which all closed geodesics are noncontractible and have no conjugate points.

1.1 Complex projective space

Consider the complex projective space $\mathbb{CP}^n$ with its standard symplectic structure ω, normalized such that a complex line has symplectic area π. Let D denote the closed unit disk. The following are the two main results of this paper.

Theorem 1.1 *Let $L \subset \mathbb{CP}^n$ be a closed Lagrangian submanifold which admits a metric of nonpositive curvature. Then there exists a smooth map $f : (D, \partial D) \to (\mathbb{CP}^n, L)$ with $f^*\omega \geq 0$ whose symplectic area satisfies*

$$0 < \int_D f^*\omega \leq \frac{\pi}{n+1}.$$

Theorem 1.2 (Audin's conjecture) *The Maslov index of the map f in Theorem 1.1 satisfies:*

(a) $\mu(f) \in \{1, 2\}$ if L is monotone;
(b) $\mu(f) = 2$ if L is a (not necessarily monotone) torus.

In particular, Theorem 1.2 (b) answers a question posed by Audin [3] in 1988: *Every Lagrangian torus in $\mathbb{C}^n$ admits a disk of Maslov index* 2. This question was answered earlier for $n = 2$ by Viterbo [64] and Polterovich [61], in the monotone case for $n \leq 24$ by Oh [59], and in the monotone case for general n by Buhovsky [14] and by Fukaya et al. [33, Theorem 6.4.35], see also Damian [24]. A different approach has been outlined by Fukaya [32]. The scheme to prove Audin's conjecture using punctured holomorphic curves was suggested by Eliashberg around 2001. The reason it took over 10 years to complete this paper are transversality problems in the non-monotone case. We solve these problems using techniques from [20].

The proof of Theorem 1.1 actually yields $n + 1$ maps $f_0, \ldots, f_n : (D, \partial D) \to (\mathbb{CP}^n, L)$, each of positive area (and Maslov index 1 or 2 in the situation of Theorem 1.2), such that $\sum_{i=0}^n \int_D f_i^*\omega \leq \pi$. We illustrate this with two examples.

1. The *Clifford torus* is the monotone torus in $\mathbb{CP}^n$ defined by

$$T^n_{\text{Clifford}} := \Big\{[z_0 : \cdots : z_n] \;\Big|\; |z_0| = \cdots = |z_n|\Big\}.$$

It admits $n + 1$ holomorphic maps $f_0, \ldots, f_n : (D, \partial D) \to (\mathbb{C}P^n, T^n_{\text{Clifford}})$ of area $\pi/(n+1)$ and Maslov index 2 given by $f_i(z) := [1 : \ldots 1 : z : 1 \ldots 1]$

with z in the ith component. Note that the boundary loops $f_i(\partial D)$ for $0 \le i \le n$ generate the first homology $H_1(T^n)$.

2. The *Chekanov torus* is an exotic monotone 2-torus T^2_{Chekanov} in $\mathbb{CP}^2$ described in [16]. We show in Appendix A that for the disks f_0, f_1, f_2 obtained in Theorem 1.1, all boundary loops $f_i(\partial D)$ represent multiples of the same homology class. This answers in the negative Viterbo's question whether for $L \cong T^n$ the boundary loops $f_i(\partial D)$ for $0 \le i \le n$ always generate the first homology $H_1(T^n)$.

1.2 A new symplectic capacity

To explore the implications of Theorem 1.1, we define a new symplectic capacity, following a suggestion by J. Etnyre. Define the *minimal symplectic area* of a Lagrangian submanifold L of a symplectic manifold (X, ω) by

$$A_{\min}(L) := \inf\left\{\int_\sigma \omega \;\middle|\; \sigma \in \pi_2(X, L),\ \int_\sigma \omega > 0\right\} \in [0, \infty].$$

Define the *Lagrangian capacity* $c_L(X, \omega)$[1] of (X, ω) to be

$$c_L(X, \omega) := \sup\{A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus}\} \in [0, \infty].$$

The Lagrangian capacity satisfies

(Monotonicity) $c_L(X, \omega) \le c_L(X', \omega')$ if there exists a symplectic embedding $\iota : (X, \omega) \hookrightarrow (X', \omega')$ with $\pi_2\big(X', \iota(X)\big) = 0$;
(Conformality) $c_L(X, \alpha\omega) = |\alpha| c_L(X, \omega)$ for $0 \ne \alpha \in \mathbb{R}$;
(Nontriviality) $0 < c_L\big(B^{2n}(1)\big)$ and $c_L\big(Z^{2n}(1)\big) < \infty$.

In particular, c_L is a (generalized) capacity in the sense of [18,44] on the class of symplectic manifolds (X, ω) with $\pi_1(X) = \pi_2(X) = 0$. Here $B^{2n}(r)$ is the open unit ball in $\mathbb{C}^n$ and $Z^{2n}(r) = B^2(r) \times \mathbb{C}^{n-1}$; open subsets of $\mathbb{C}^n$ are always equipped with the canonical symplectic structure. The first two properties are obvious. The property $0 < c_L\big(B^{2n}(1)\big)$ holds because the unit ball contains a small *standard torus* $T^n(r) := S^1(r) \times \cdots \times S^1(r)$, where $S^1(r) \subset \mathbb{C}$ is the sphere of radius r. For the last property, recall Chekanov's result [15] that

$$A_{\min}(L) \le d(L)$$

for every closed Lagrangian submanifold L in $\mathbb{C}^n$. Here $d(A)$ is the *displacement energy* of a subset $A \subset \mathbb{C}^n$, i.e., the minimal Hofer energy of a compactly

[1] The "L" in c_L just stands for "Lagrangian" and does not refer to a particular Lagrangian submanifold L.

supported Hamiltonian diffeomorphism that displaces A from itself (see [44]). This implies

$$c_L(U) \le d(U)$$

for every open subset $U \subset \mathbb{C}^n$. Since the symplectic cylinder $Z^{2n}(1)$ has displacement energy π and contains the standard torus $T^n(1)$ of minimal symplectic area π, it follows that

$$c_L\big(Z^{2n}(1)\big) = \pi.$$

Most known symplectic capacities (exceptions being the higher Ekeland–Hofer capacities [27] for subsets of $\mathbb{C}^n$ and the ECH capacities [46] in dimension 4) take the same value on the unit ball and on the cylinder. Surprisingly, this is not the case for the Lagrangian capacity:

Corollary 1.3

$$c_L\big(B^{2n}(1)\big) = \frac{\pi}{n}.$$

Proof The standard torus $T^n(r)$ is contained in the unit ball if and only if $r < 1/\sqrt{n}$. This proves one inequality. For the converse inequality, suppose that the unit ball contains a Lagrangian torus L with $A_{\min}(L) \ge \pi/n$. After replacing L by αL for a suitable $0 < \alpha \le 1$ we may assume that $A_{\min}(L) = \pi/n$. Compactifying the closed unit ball to $\mathbb{CP}^n$, L gives rise to a Lagrangian torus L' in $\mathbb{CP}^n$ with the property that all disks with boundary on L' have symplectic area a multiple of π/n. (This property is clear for disks contained in the open ball $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$; for a disk passing through $\mathbb{CP}^{n-1}$ it follows by gluing it along its boundary with a disk in the ball to a sphere whose area is a multiple of π.) But this property contradicts Theorem 1.1. □

For an application of this result, consider a *polydisk* $P^{2n}(r) := B^2(r) \times \cdots \times B^2(r)$. It contains the standard torus $T^n(r)$ and is contained in the cylinder $Z^{2n}(r)$, hence

$$c_L\big(P^{2n}(r)\big) = \pi r^2.$$

Now Monotonicity of the Lagrangian capacity implies the following non-squeezing result due to Ekeland and Hofer (Fig. 1).

Corollary 1.4 (Ekeland and Hofer [27]) *The polydisk $P^{2n}(r)$ can be symplectically embedded into the ball $B^{2n}(1)$ if and only if*

$$r \le \frac{1}{\sqrt{n}}.$$

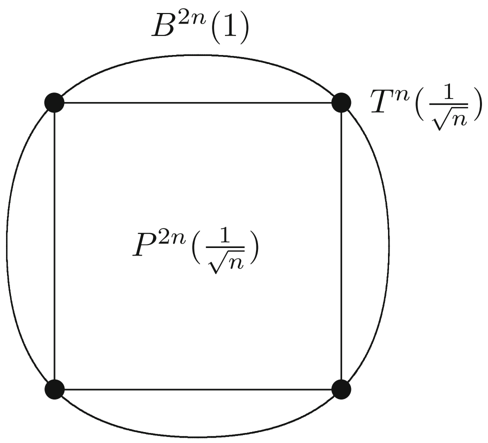

Fig. 1 A polydisk inscribed in a ball

We end this paragraph with a conjecture for the Lagrangian capacity of an *ellipsoid*

$$E(a_1, \ldots, a_n) := \left\{ z \in \mathbb{C}^n \,\middle|\, \frac{|z_1|^2}{a_1} + \cdots + \frac{|z_n|^2}{a_n} \leq 1 \right\}$$

with $0 < a_1 \leq \cdots \leq a_n < \infty$.

Conjecture 1.5 The Lagrangian capacity of an ellipsoid is given by

$$c_L\Big(E(a_1, \ldots, a_n)\Big) = \frac{\pi}{1/a_1 + \cdots + 1/a_n}.$$

It is shown in [18] that this conjecture would imply $c_L = \lim_{k\to\infty} \frac{1}{k} c_k^{\mathrm{EH}}$ on all ellipsoids, where c_k^{EH} denotes the kth Ekeland–Hofer capacity [27]. It would be interesting to see whether this relation continues to hold on more general subsets of $\mathbb{R}^{2n}$, e.g. on convex sets. The relation may also be compared to a similar formula recovering the volume of a 4-dimensional Liouville domain from its ECH capacities [23].

Remark 1.6 If we define the Lagrangian capacity c_L using arbitrary closed Lagrangian submanifolds instead of just tori we still obtain a capacity with the property $c_L(U) \leq d(U)$ for $U \subset \mathbb{C}^n$. The Lagrangian capacity of the unit ball is still $\geq \pi/n$, but we do not know whether equality holds for $n > 2$. For $n = 2$, equality holds in view of Theorem 1.1 (and the proof of Corollary 1.3) because all closed Lagrangian submanifolds of $\mathbb{C}^2$ admit a metric of nonpositive curvature.

1.3 Extremal Lagrangian tori

Let us call a Lagrangian torus L in a symplectic manifold (X, ω) *extremal* if it maximizes $A_{\min}$, i.e., $A_{\min}(L) = c_L(X, \omega)$. Recall that L is *monotone* if its Maslov class is positively proportional to the symplectic area class on $\pi_2(X, L)$, i.e., $\mu = 2a[\omega]$ for some $a > 0$. This implies that (X, ω) is monotone with $c_1(TX) = a[\omega]$ on $\pi_2(X)$ (see e.g. [17]). In particular, for $X = \mathbb{CP}^n$ with its standard symplectic form we have $a = (n+1)/\pi$, so for a monotone Lagrangian torus $L \subset \mathbb{CP}^n$ the values of $[\omega]$ on $\pi_2(\mathbb{CP}^n, L)$ are integer multiples $\pi/(n+1)$ (since all Maslov indices are even). Therefore, Theorem 1.1 implies

Corollary 1.7 *Every monotone Lagrangian torus in* $\mathbb{CP}^n$ *is extremal.*

Conjecture 1.8 Every extremal Lagrangian torus in $\mathbb{CP}^n$ is monotone.

Turning to a different manifold, the following conjecture is motivated by a question of L. Lazzarini.

Conjecture 1.9 Every extremal Lagrangian torus in the closed unit ball $\overline{B}^{2n}(1)$ is entirely contained in the boundary $\partial B^{2n}(1)$.

Remark 1.10 Note that the standard torus $T^n(1/\sqrt{n}) \subset \partial B^{2n}(1)$ is extremal. The following fact, which is proved in Appendix B, lends some plausibility to the conjecture. It establishes a phenomenon which is called "non-removable intersection" in [60]:

For a closed Lagrangian submanifold $L \subset \partial B^{2n}(1)$ *no point can be pushed into the interior by a Hamiltonian isotopy without making* L *exit the closed ball at some other point.*

Remark 1.11 One motivation for our interest in monotone Lagrangian submanifolds is the observation (see e.g. [17]) that minimal (i.e., of zero mean curvature) Lagrangian submanifolds in $\mathbb{CP}^n$ are monotone.

1.4 The chord conjecture

Another application concerns *Arnold's chord conjecture* [2]. Let $U \subset \mathbb{C}^n$ be a bounded star-shaped domain (with respect to the origin) with smooth boundary $S = \partial U$. The 1-form

$$\lambda := \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

on $\mathbb{C}^n$ induces a contact form on S, and every contact form on the sphere S^{2n-1} defining the standard contact structure arises in this way (see Sect. 2 for the basic definitions concerning contact manifolds). A *Reeb chord of length* T to a Legendrian submanifold $\Lambda \subset S$ is an orbit $\gamma : [0, T] \to S$ of the Reeb vector field with $\gamma(0), \gamma(T) \in \Lambda$. The chord conjecture states that every closed Lagrangian submanifold of S possesses a Reeb chord. It was proved by the second author in [57].

To see the relation to the Lagrangian capacity, let us recall the proof in [57]. Suppose that there exists no Reeb chord of length $\leq T$. Then we can construct a Lagrangian submanifold $L \subset \mathbb{C}^n$ out of Λ as follows. Move Λ in S by the Reeb flow until time T, then push it down radially in $\mathbb{C}^n$ to the sphere εS, $\varepsilon > 0$, move it in εS by the backward Reeb flow until time $-T$ and push it up radially to S. Smoothing corners, this yields a Lagrangian submanifold $L \subset \mathbb{C}^n$ diffeomorphic to $S^1 \times \Lambda$ with minimal symplectic area

$$A_{\min}(L) = (1 - \varepsilon)T.$$

Since $A_{\min}(L) \leq d(L) \leq d(U)$ by Chekanov's theorem, this proves the result in [57]: *Every closed Legendrian submanifold Λ in $S = \partial U$ possesses a Reeb chord of length $T \leq d(U)$*. If Λ admits a metric of nonpositive curvature the above argument yields the following sharper estimate.

Corollary 1.12 *Every closed Legendrian submanifold Λ of nonpositive curvature in the boundary S of a star-shaped domain $U \subset \mathbb{C}^n$ possesses a Reeb chord of length*

$$T \leq c_L(U).$$

Let us compare the two estimates for the unit sphere S in $\mathbb{C}^n$. Here all Reeb orbits are closed of length π and form the fibres of the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$. The estimate in [57] yields a Reeb chord of length $T \leq \pi$, which may just be a closed Reeb orbit meeting Λ once. The estimate in Corollary 1.12 combined with Corollary 1.3 yields

Corollary 1.13 *Every closed Legendrian submanifold Λ of nonpositive curvature in the unit sphere $S \subset \mathbb{C}^n$, $n \geq 2$, possesses a Reeb chord of length*

$$T \leq \frac{\pi}{n}.$$

In particular, Λ meets some fibre of the Hopf fibration at least twice.

In [69] F. Ziltener has shown that the last statement actually holds without the assumption of nonpositive curvature. This follows from the proof in [57] described above and the observation that the displacement energy of the Lagrangian submanifold $L \cong S^1 \times \Lambda$ is strictly smaller than π.

The last statement can be rephrased as follows. Call a Lagrangian submanifold $L \subset \mathbb{CP}^n$ *exact* if every disk with boundary on L has symplectic area an integer multiple of π. Every exact Lagrangian submanifold $L \subset \mathbb{CP}^n$

lifts to a Legendrian submanifold $\Lambda \subset S^{2n+1}$ that intersects each fibre of the Hopf fibration at most once. (A horizontal lift of L exist locally since L is Lagrangian; the obstructions to a global horizontal lift are the holonomies of the connection form 2λ along loops in L, which vanish because the integral of the curvature 2ω over each disk with boundary on L is an integer multiple of 2π.) Hence the preceding discussion implies

Corollary 1.14 ([69]) *There exists no exact closed Lagrangian submanifold in* $\mathbb{CP}^n$.

For $n = 1$ this is obvious (and was the motivation for Arnold's conjecture in [2]). For $n > 1$ it generalizes (for manifolds of nonpositive curvature) Gromov's result [37] that there is no closed exact Lagrangian submanifold in $\mathbb{C}^n$. However, Gromov's original technique does not seem applicable here.

The definition of a Reeb chord given above includes periodic orbits intersecting Λ. Let us call a Reeb chord *honest* if it is not a closed Reeb orbit meeting Λ just once. Arnold's original conjecture asks for the existence of honest Reeb chords. Corollary 1.12 can be used to prove this stronger conjecture. For example, if $S \subset \mathbb{C}^n$ is C^1-close to the unit sphere, then all closed Reeb chords have length at least $\pi - \varepsilon$, whereas Corollary 1.12 yields a Reeb chord of length at most $\pi/n + \varepsilon$. Hence we have

Corollary 1.15 *Let $S \subset \mathbb{C}^n$, $n \geq 2$, be sufficiently C^1-close to the unit sphere. Then every closed Legendrian submanifold Λ of nonpositive curvature in S possesses an honest Reeb chord.*

1.5 Lagrangian submanifolds and symplectic balls

Biran [8] has shown that a symplectically embedded ball $B \subset \mathbb{CP}^n$ of radius $r > 1/\sqrt{2}$ must intersect $\mathbb{R}P^n \subset \mathbb{CP}^n$. This turns out to be related to a quite general intersection phenomenon.

Theorem 1.16 *Let $L \subset \mathbb{CP}^n$ be a closed Lagrangian submanifold which admits a metric without contractible geodesics (e.g. one of nonpositive curvature). Let $B \subset \mathbb{CP}^n$ be a symplectically embedded ball of radius r. If $L \cap B = \emptyset$, then*

$$A_{\min}(L) + \pi r^2 \leq \pi.$$

Note that $A_{\min}(\mathbb{RP}^n) = \pi/2$, so the estimate in Theorem 1.16 would imply Biran's result. However, the theorem is not applicable to $L = \mathbb{RP}^n$ because every metric on $\mathbb{R}P^n$ has contractible geodesics. Combined with Corollary 1.7, Theorem 1.16 implies that every monotone Lagrangian torus in $\mathbb{CP}^n$ must intersect every embedded ball of radius $r > \sqrt{\frac{n}{n+1}}$; see Biran and Cornea [9] for generalizations of this result.

1.6 Uniruled symplectic manifolds

Let us call a closed symplectic manifold (X, ω) *uniruled* if it has some nonvanishing Gromov–Witten invariant of holomorphic spheres passing through a point (plus additional constraints). Ruan [63] has proved that on a Kähler manifold this is equivalent to the algebro-geometric definition that X is covered by rational curves. For example, every Fano manifold is uniruled (see [49]). The following result is due to Viterbo [65] for "strongly Fano manifolds", and to Eliashberg, Givental and Hofer [28] in the general case. We include its proof in Sect. 4.

Theorem 1.17 ([28,65]) *Let L be a closed manifold of dimension ≥ 3 which carries a metric of negative curvature. Then L admits no Lagrangian embedding into a uniruled symplectic manifold.*

Remark 1.18 In dimension 2 the result is not true because of Givental's nonorientable Lagrangian surfaces in $\mathbb{C}^2$ (see [4]). But we still obtain the following restriction: a closed orientable surface of genus ≥ 2 admits no Lagrangian embedding into a uniruled symplectic 4-manifold. This is an easy consequence of McDuff's theorem (implicitly contained in [55]) that every uniruled symplectic 4-manifold has $b_2^+ = 1$, see [66, Section 2.2] and [68, Proposition 2.17].

Remark 1.19 The proof of Theorem 1.1 yields for every closed Lagrangian surface $L \subset \mathbb{CP}^2$ which carries a metric of negative curvature (and is therefore nonorientable) the estimate $A_{\min}(L) \leq \pi/6$.

In [58], J. Nash proved that every compact smooth manifold is the real part of a real algebraic manifold (i.e., a smooth projective variety defined by real equations in $\mathbb{CP}^N$). In the same paper he conjectured that the real algebraic manifold can be chosen to be birational to $\mathbb{CP}^n$. Now the real part of a real algebraic manifold is Lagrangian (see [65]), and an algebraic manifold birational to $\mathbb{CP}^n$ is uniruled (it is even rationally connected, see [51]) and simply connected (see [35]). Hence Theorem 1.17 and Remark 1.18 imply

Corollary 1.20 ([21,28,50,65]) *The Nash conjecture is false in all dimensions ≥ 2. More precisely, we have:*

(a) For $n \geq 3$, the real part of a real algebraic manifold birational to $\mathbb{CP}^n$ carries no metric of negative curvature.
(b) The real part of a real algebraic surface birational to $\mathbb{CP}^2$ is either the sphere, the torus, or a nonorientable surface.

For $n = 2$ the result is due to Comessatti [21] and it is sharp: the sphere and the torus occur in quadrics $x^2 + y^2 \pm z^2 = t^2$, and the nonorientable surfaces occur in blow-ups of $\mathbb{CP}^2$ at real points. Already for $n = 3$ the result is far from optimal. In fact, Kollár has derived a short list of possible topological types for real parts of real algebraic threefolds [50,51].

Now suppose that the values of the symplectic form ω on $\pi_2(X)$ are given by $k\,a$, $k \in \mathbb{Z}$, for some $a > 0$. Let us call (X, ω) *minimally uniruled* if the holomorphic spheres in the definition of "uniruled" have symplectic area a. Call a Lagrangian embedding $L \hookrightarrow X$ *exact* if every disk with boundary on L has symplectic area an integer multiple of a. With these notations, Corollary 1.14 can be generalized as follows.

Theorem 1.21 *Let L be a closed manifold which carries a metric without contractible geodesics. Then L admits no exact Lagrangian embedding into a minimally uniruled symplectic manifold.*

Examples of minimally uniruled symplectic manifolds are $\mathbb{CP}^n$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ with their standard symplectic structures. The real locus $\mathbb{R}P^n \subset \mathbb{CP}^n$ is not exact, while the antidiagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is exact.

1.7 Size of Weinstein neighbourhoods

Weinstein's Lagrangian neighbourhood theorem states that any Lagrangian submanifold $L \subset (X, \omega)$ has a tubular neighbourhood which is symplectomorphic to a neighbourhood of the zero section in the cotangent bundle T^*L with the canonical symplectic form. We are interested in the maximal "size" of such a neighbourhood, which can be quantified as follows. For a Riemannian manifold (Q^n, g) let $D^*(Q, g) := \{(q, p) \in T^*Q \mid \|p\|_g \leq 1\}$ be the unit codisk bundle. Fix a symplectic manifold (X^{2n}, ω) and define the *embedding capacity*

$$c^{(X,\omega)}(Q, g) := \inf\{\alpha > 0 \mid D^*(Q, g) \hookrightarrow (X, \alpha\omega)\},$$

where $\hookrightarrow$ means symplectic embedding with respect to the canonical symplectic structure on T^*Q. This is the embedding capacity of $D^*(Q, g)$ into (X, ω) in the sense of [18]. It has the following elementary properties with respect to the metric:

(Invariance) $c^{(X,\omega)}(Q, g) = c^{(X,\omega)}(Q', g')$ if (Q, g) and (Q', g') are isometric;
(Monotonicity) $c^{(X,\omega)}(Q, g) \leq c^{(X,\omega)}(Q, h)$ if $g \leq h$;
(Conformality) $c^{(X,\omega)}(Q, \lambda g) = \lambda c^{(X,\omega)}(Q, g)$ for $\lambda > 0$;
(Nontriviality) $c^{(X,\omega)}(Q, g) < \infty$ if Q admits Lagrangian embeddings into (X, ω), and $c^{(X,\omega)}(Q, g) > 0$ if (X, ω) has finite volume.

For the last property, note that a symplectic embedding $D^*(Q, g) \hookrightarrow (X, \alpha\omega)$ yields

$$\mathrm{vol}(Q, g)\mathrm{vol}B^n(1) = \mathrm{vol}D^*(Q, g) \leq \mathrm{vol}(X, \alpha\omega) = \alpha^n \mathrm{vol}(X, \omega)$$

and hence the lower estimate

$$c^{(X,\omega)}(Q, g) \geq \sqrt[n]{\frac{\mathrm{vol}(Q, g)\mathrm{vol}B^n(1)}{\mathrm{vol}(X, \omega)}}. \tag{1}$$

Let $\ell_{\min}(Q, g)$ denote the minimal length of a closed geodesic on (Q, g). Then the methods of this paper yield the following lower estimate.

Theorem 1.22 *The embedding capacity of a Riemannian manifold (Q, g) of nonpositive curvature into $\mathbb{CP}^n$ with its standard symplectic form satisfies*

$$c^{\mathbb{CP}^n}(Q, g) \geq \frac{(n+1)\ell_{\min}(Q, g)}{\pi}.$$

Some explicit computations in Appendix C show:

1. For the flat torus $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, the estimate in Theorem 1.22 is better than the volume estimate (1) for each $n > 1$. Moreover, the volume bound grows as $\sqrt{n}$, while the bound in Theorem 1.22 grows linearly in n.

2. Explicit embeddings of the flat torus $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ into $\mathbb{C}P^n$ yield an upper bound $c^{\mathbb{CP}^n}(T^n) \leq 2(n + \sqrt{n})$, which is stricly larger than the lower bound $2(n + 1)$ from Theorem 1.22 for each $n > 1$. This suggests that the estimate in Theorem 1.22 may not be sharp. Note, however, that the quotient of the upper and lower bound tends to 1 as $n \to \infty$.

Remark 1.23 (a) The techniques of this paper work well for Lagrangian submanifolds which admit metrics of nonpositive curvature, or at least without contractible geodesics. On the other hand, traditional techniques such as Gromov's holomorphic disks and Floer homology work well for simply connected Lagrangians. Can one combine these techniques? A test case could be the product of a manifold of positive with a manifold of negative curvature.
(b) Neck stretching techniques were also used in [52,66] to study Lagrangian embeddings in dimensions 4 and 6.

2 Punctured holomorphic curves

In this section we recall the definitions and basic properties of punctured holomorphic curves, see [12,19,28] for details.

2.1 Contact manifolds

A *contact form* on a $(2n-1)$–dimensional manifold M^{2n-1} is a 1-form λ for which $\lambda \wedge (d\lambda)^{n-1}$ is a volume form. Its *Reeb vector field* is the unique vector field R_λ satisfying $i_{R_\lambda} d\lambda = 0$ and $\lambda(R_\lambda) = 1$. The distribution $\xi := \ker \lambda$ is called a *(cooriented) contact structure*, and the pair (M, ξ) a *contact manifold*.

Consider a closed orbit γ of the Reeb vector field. The linearized Reeb flow along γ preserves the contact distribution ξ, and the maps $\Psi_t : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ preserve $d\lambda$. Let $\Phi : \xi|_\gamma \xrightarrow{\cong} S^1 \times \mathbb{C}^{n-1}$ be a trivialization of ξ along γ. Then $\Phi_t \Psi_t \Phi_0^{-1}$ is a path of symplectic matrices which has a *Conley–Zehnder index* (defined in [22] if γ is *nondegenerate*, i.e., the linearized return map on a transverse section has no eigenvalue 1, and in [64] in general) and a *Robbin–Salamon index* (defined in [62]). We call

$$\mathrm{CZ}(\gamma, \Phi) := \mathrm{CZ}\left(\Phi_t \Psi_t \Phi_0^{-1}\right) \in \mathbb{Z}, \qquad \mathrm{RS}(\gamma, \Phi) := \mathrm{RS}\left(\Phi_t \Psi_t \Phi_0^{-1}\right) \in \frac{1}{2}\mathbb{Z}$$

the Conley–Zehnder (or Robbin–Salamon, respectively) index of γ with respect to the trivialization Φ.

Following [11,12], we say that (M, λ) is *Morse–Bott* if the following holds: For all $T > 0$ the set $N_T \subset M$ formed by the T-periodic Reeb orbits is a closed submanifold, the rank of $d\lambda|_{N_T}$ is locally constant, and $T_p N_T = \ker(T_p \phi_T - \mathbb{1})$ for all $p \in N_T$, where ϕ_t is the Reeb flow. Dividing out time reparametrization, we obtain the orbifold of unparametrized Reeb orbits $\Gamma_T := N_T / S^1$. Note that the case $\dim \Gamma_T = \dim N_T - 1 = 0$ corresponds to nondegeneracy of closed Reeb orbits. We will refer to this case as the *Morse case*. In the Morse–Bott case, the Conley–Zehnder and Robbin–Salamon indices are constant along each connected component Γ of Γ_T and we denote them by $\mathrm{CZ}(\Gamma)$ or $\mathrm{RS}(\Gamma)$, respectively. They are related by

$$\mathrm{CZ}(\Gamma) = \mathrm{RS}(\Gamma) - \frac{1}{2} \dim \Gamma. \tag{2}$$

An important special case is the *geodesic flow* on the unit cotangent bundle $M = S^*L$ of a Riemannian manifold (L, g). This is the Reeb flow of the contact form $\lambda = p\,dq$ in canonical coordinates (q, p) on T^*L. A closed geodesic q on L lifts to the closed Reeb orbit $\gamma(t) = \big(q(t), \dot q(t)\big)$. Here we use the identification $TL \cong T^*L$ via $v \mapsto g(v, .)$ without further notice. A trivialization Φ of the contact distribution ξ along γ is equivalent to a trivialization of $T(T^*L)$ as a symplectic vector bundle along q (viewed as a path in the zero section) which is still denoted by Φ. Note that the zero section $L \subset T^*L$ is Lagrangian for $d\lambda$. Let

$$\mu(\gamma, \Phi) := \mu(q, \Phi) := \mu(\Phi_t T_{q(t)} L)$$

be the *Maslov index* of the induced loop of Lagrangian subspaces of $\mathbb{C}^n$ (see e.g. [4]). Now $\mathrm{CZ}(\gamma, \Phi)$ and $\mu(\gamma, \Phi)$ transform in the opposite way under change of the trivialization. So their sum is independent of Φ, and in fact is a well-known quantity:

Lemma 2.1 (Viterbo [64]) *For the lift γ of a (possibly degenerate) closed geodesic q,*

$$\mathrm{CZ}(\gamma, \Phi) + \mu(\gamma, \Phi) = \mathrm{ind}(\gamma),$$

where $\mathrm{ind}(\gamma) := \mathrm{ind}(q)$ *is the Morse index of q.*

We will need the following lemma about Morse indices:

Lemma 2.2 *Let L^n be a closed manifold which admits a metric of nonpositive sectional curvature. Then for every $c > 0$ there exists a metric on L such that every closed geodesic q of length $\le c$ is noncontractible and nondegenerate and satisfies*

$$0 \le \mathrm{ind}(q) \le n - 1.$$

In a metric of negative curvature every closed geodesic is noncontractible and nondegenerate of index zero.

Proof By the Morse Index Theorem for closed geodesics (see e.g. [48], Theorem 2.5.14),

$$\mathrm{ind}(q) = \mathrm{ind}_\Omega(q) + \mathrm{concav}(q),$$

where $\mathrm{ind}_\Omega(q)$ equals the number of conjugate points along q, and the nullity and concavity of q satisfy $\mathrm{null}(q) + \mathrm{concav}(q) \le n - 1$. Now a metric g_0 of nonpositive curvature has no conjugate points along any geodesic (see e.g [48]., Theorem 2.6.2), hence $\mathrm{ind}(q) + \mathrm{null}(q) \le n - 1$ for every closed g_0-geodesic.

Consider the space of all closed g_0-geodesics of length $\le c$, parametrized with constant speed over the time interval [0, 1]. Since this space is compact (say, with the C^2 topology), there exists a C^2-neighbourhood $\mathcal{U}$ of g in the space of metrics such that, for every $g \in \mathcal{U}$ and every closed g-geodesic q of length $\le c$, we still have $\mathrm{ind}(q) + \mathrm{null}(q) \le n - 1$. Now there exist metrics $g \in \mathcal{U}$ for which all closed geodesics q of length $\le c$ are nondegenerate (see e.g. [47], Theorem 3.3.9), and therefore $\mathrm{ind}(q) \le n - 1$.

By Hadamard's Theorem (see e.g. [48], Theorem 2.6.6), all closed g_0-geodesics are noncontractible; this persists for g-geodesics of length $\le c$ for $g \in \mathcal{U}$. Finally, if g_0 has negative curvature, then the nullity and concavity are zero, so every closed g_0-geodesic is nondegenerate of index 0. □

2.2 Symplectic cobordisms

The *symplectization* of a contact manifold (M, ξ) is the symplectic manifold $\big(\mathbb{R} \times M, d(e^r\lambda)\big)$, where r is the coordinate on $\mathbb{R}$. A *symplectic cobordism* is a symplectic manifold (X, ω) whose ends are the positive or negative halves of symplectizations. This means that for a some compact subset $X_0 \subset X$ and contact manifolds $(\overline{M}, \overline{\lambda})$ $(\underline{M}, \underline{\lambda})$, we have a symplectomorphism

$$(X \backslash X_0, \omega) \cong \big(\mathbb{R}_+ \times \overline{M}, d(e^r\overline{\lambda})\big) \amalg \big(\mathbb{R}_- \times \underline{M}, d(e^r\underline{\lambda})\big).$$

Here we allow one or both of $\overline{M}$, $\underline{M}$ to be empty. The ends modelled over $\overline{M}$ and $\underline{M}$ are called the *positive and negative ends*, respectively. Obvious examples of symplectic cobordisms are closed symplectic manifolds and symplectizations.

Example 2.3 Cotangent bundles are symplectic cobordisms as follows. Let $M = S^*L$ be the unit cotangent bundle of a Riemannian manifold (L, g), with the contact form $\lambda = p\,dq|_M$. We identify L with the zero section in T^*L. Then the map $\mathbb{R}\times M \to T^*L\backslash L, (r, q, p) \mapsto (q, e^r p)$ yields a symplectomorphism

$$\big(\mathbb{R} \times M, d(e^r\lambda)\big) \cong (T^*L\backslash L, dp\,dq).$$

This shows that the cotangent bundle $(T^*L, dp\,dq)$ is a symplectic cobordism with one positive end modelled over (M, λ).

2.3 Almost complex structures

An almost complex structure J on a symplectization $\big(\mathbb{R}\times M, d(e^r\lambda)\big)$ is called *compatible with* λ if it is translation invariant in the $\mathbb{R}$-direction, leaves the contact structure ξ invariant, maps $\partial/\partial r$ to the Reeb vector field R_λ, and $d(e^r\lambda)(\cdot, J\cdot)$ defines a Riemannian metric. An almost complex structure on a symplectic cobordism (X, ω) is called *compatible* if it is compatible with the contact forms $\overline{\lambda}$, $\underline{\lambda}$ outside some compact set and $\omega(\cdot, J\cdot)$ defines a Riemannian metric. Compatible almost complex structures exist and form a contractible space $\mathcal{J}$ (with the C^∞ topology).

2.4 Punctured holomorphic curves

Consider a symplectization $\big(\mathbb{R} \times M, d(e^r\lambda)\big)$ with a compatible almost complex structure J. Let $\gamma : [0, T] \to M$ be a (not necessarily simple) closed orbit of the Reeb vector field R_λ of period T. A J-holomorphic map $f = (a, u) : D\backslash 0 \to \mathbb{R} \times M$ of the punctured unit disk is called *positively (or negatively)* asymptotic to γ if $\lim_{\rho\to 0} a(\rho e^{i\theta}) = \infty$ (or $-\infty$) and $\lim_{\rho\to 0} u(\rho e^{i\theta}) = \gamma(T\theta/2\pi)$ (or $\gamma(-T\theta/2\pi)$, respectively) uniformly in θ.

A *punctured holomorphic curve* in (X, J) consists of the following data:

- A Riemann surface (Σ, j) with distinct positive and negative points $\overline{z} := (\overline{z}_1, \dots, \overline{z}_{\overline{p}})$, $\underline{z} := (\underline{z}_1, \dots, \underline{z}_{\underline{p}})$. We denote by $\dot{\Sigma} := \Sigma \setminus \{\overline{z}_i, \underline{z}_j\}$ the corresponding punctured Riemann surface.
- Corresponding vectors $\overline{\Gamma} := (\overline{\gamma}_1, \dots, \overline{\gamma}_{\overline{p}})$, $\underline{\Gamma} := (\underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{p}})$ of closed Reeb orbits in $\overline{M}$, $\underline{M}$.
- A (j, J)–holomorphic map $f : \dot{\Sigma} \to X$ which is positively (or negatively) asymptotic to $\overline{\gamma}_j$ (or $\underline{\gamma}_j$) at the punctures $\overline{z}_j$ (or $\underline{z}_j$, respectively).

In a symplectization $\mathbb{R} \times M$, a *cylinder* over a T-periodic Reeb orbit γ,

$$f : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times M, \qquad (s, t) \mapsto \big(Ts, \gamma(Tt)\big),$$

is a punctured holomorphic curve with one positive and one negative puncture.

Denote by $\bar{\Sigma}$ the compactification of the punctured surface $\dot{\Sigma}$ by adding a circle at each puncture, and by $\bar{X}$ the compactification of the manifold X by adding a copy of $\overline{M}$ or $\underline{M}$ at the positive or negative end, respectively. In view of the behaviour near the punctures, the holomorphic map $f : \dot{\Sigma} \to X$ above extends to a continuous map $\bar{f} : \bar{\Sigma} \to \bar{X}$. This extension represents a relative homology class

$$[\bar{f}] \in H_2(\bar{X}, \overline{\Gamma} \cup \underline{\Gamma}).$$

Example 2.4 The cotangent bundle T^*L of a Riemannian manifold (L, g) carries a canonical almost complex structure J_g induced by the Riemannian metric. This structure leaves the cylinders $\{\big(q(t), s\,\dot{q}(t)\big) \mid s, t \in \mathbb{R}\}$ over closed geodesics q invariant. Now J_g is not compatible according to the definition above. But it can be deformed to a compatible almost complex structure J which still leaves all the cylinders over closed geodesics invariant. Then these cylinders, appropriately parametrized, are punctured holomorphic curves in T^*L with two positive punctures.

2.5 Neck stretching

Consider a closed connected symplectic manifold (X, ω) and a closed hypersurface $M \subset X$ of *contact type*. This means that M carries a contact form λ such that $d\lambda = \omega|_M$. Let J_M be an almost complex structure on $\mathbb{R} \times M$ which is compatible with λ. A neighbourhood of M is symplectomorphic to $\big([-\varepsilon, \varepsilon] \times M, d(e^r\lambda)\big)$. Assume that $X \setminus M = X_0^+ \coprod X_0^-$ consists of two components with $\{\pm\epsilon\} \times M \subset X_0^{\pm}$. Pick a compatible almost complex structure J on (X, ω) whose restriction to $[-\varepsilon, \varepsilon] \times M$ coincides with J_M.

Define a 1-parameter family of symplectic manifolds as follows. For a real number $k \geq 0$ let

$$X_k := X_0^- \cup_M [-k, 0] \times M \cup_M X_0^+.$$

This manifold is, of course, canonically diffeomorphic to X, but X_k is a more convenient domain for describing the deformed structures. Define an almost complex structure on X_k by

$$J_k := \begin{cases} J & \text{on } X_0^\pm, \\ J_M & \text{on } [-k, 0] \times M. \end{cases}$$

This almost complex structure is compatible with the symplectic form on X_k given by

$$\omega_k := \begin{cases} \omega & \text{on } X_0^+, \\ d(e^r \lambda) & \text{on } [-k, 0] \times M, \\ e^{-k}\omega & \text{on } X_0^-. \end{cases}$$

In the limit $k \to \infty$ we obtain three symplectic cobordisms: $X^+ := X_0^+ \cup_M \mathbb{R}_- \times M$ (with one negative end), $X^- := X_0^- \cup_M \mathbb{R}_+ \times M$ (with one positive end), and the symplectization $\mathbb{R} \times M$. Here $\mathbb{R} \times M$ is equipped with the symplectic form $d(e^r\lambda)$ and almost complex structure J_M, and $X^\pm$ are equipped with the symplectic forms $\omega^\pm$ and almost complex structures $J^\pm$ satisfying $\omega^\pm = \omega$ on $X_0^\pm$, $\omega^\pm = d(e^r\lambda)$ on $\mathbb{R}_\pm \times M$, $J^\pm = J$ on $X_0^\pm$, $J^\pm = J_M$ on $\mathbb{R}_\pm \times M$.

Example 2.5 We will apply the neck stretching construction in the following situation. Let L be a closed Lagrangian submanifold of (X, ω). Pick a Riemannian metric on L. After rescaling the metric by a small constant, a tubular neighbourhood of L is symplectomorphic to $W := \{(q, p) \in T^*L \mid |p| \leq 1\}$ with the canonical symplectic form $dp\,dq$. The boundary $M := \partial W$ is a contact type hypersurface in X with contact form $\lambda := p\,dq|_M$. It decomposes X into two components $X_0^- = \operatorname{int} W$ and $X_0^+ = X \backslash W$. So we can stretch the neck along M as described above. In this case, the limiting manifolds can be identified as $(X^+, \omega^+) \cong (X \backslash L, \omega)$, $(X^-, \omega^-) \cong (T^*L, dp\,dq)$, and $\big(\mathbb{R} \times M, d(e^r\lambda)\big) \cong (T^*L \backslash L, dp\,dq)$.

Lemma 2.6 *Consider a Riemannian manifold (L, g) and a symplectic embedding $D^*L \hookrightarrow X$ of its unit cotangent bundle into a closed symplectic manifold (X, ω). Let J be a compatible almost complex structure on $X \backslash L$ which is cylindrical on $D^*L \backslash L$, and $f : \mathbb{C} \to X \backslash L$ be a holomorphic plane negatively* *asymptotic to the closed Reeb orbit* $(\gamma, \dot{\gamma})$ *corresponding to a closed geodesic* γ*. Then the length of* γ *satisfies*

$$\ell(\gamma) \leq \int_f \omega.$$

Proof The map $(r, q, p) \mapsto (q, e^r p)$ defines a symplectomorphism $(\mathbb{R} \times S^*L, d(e^r\lambda)) \to (D^*L \setminus L, \omega)$ with the contact form $\lambda = p\,dq|_{S^*L}$. Denote by $\hat{\gamma}$ the lift of the closed geodesic γ to S^*L. Then positivity of ω and $d\lambda$ on J-holomorphic curves implies

$$\int_f \omega \geq \int_{f \cap D^*L} d(e^r\lambda) = \int_{f \cap S^*L} \lambda \geq \int_{\hat{\gamma}} \lambda = \ell(\gamma).$$

□

2.6 Broken holomorphic curves

Now we will describe the limiting objects of sequences of J_k-holomorphic curves in the neck stretching procedure, following [19]. We retain the setup of the preceding section. For an integer $N \geq 2$ set

$$(X^{(\nu)}, \omega^{(\nu)}, J^{(\nu)}) := \begin{cases} (X^-, \omega^-, J^-) & \text{for } \nu = 1, \\ (\mathbb{R} \times M, d(e^r\lambda), J_M) & \text{for } \nu = 2, \dots, N-1, \\ (X^+, \omega^+, J^+) & \text{for } \nu = N. \end{cases}$$

Define the *split symplectic cobordism*

$$X^* := \coprod_{\nu=1}^{N} X^{(\nu)},$$

equipped with the symplectic and almost complex structures ω^*, J^* induced by the $\omega^{(\nu)}$, $J^{(\nu)}$. Glue the positive boundary component of the compactification $\overline{X^{(\nu)}}$ (by copies of M) to the negative boundary component of $\overline{X^{(\nu+1)}}$ to obtain a compact topological space

$$\bar{X} := \overline{X^{(1)}} \cup_M \cdots \cup_M \overline{X^{(N)}}.$$

Note that $\bar{X}$ is naturally homeomorphic to X (see the proof of Lemma 2.8 below for a particular homeomorphism), so we can identify homology classes in X and $\bar{X}$.

Let Σ be a closed oriented surface with q distinct points $\mathbf{z} = (z^1, \ldots, z^q)$, and $\Delta = \Delta_n \cup \Delta_p \subset \Sigma \setminus \{z^1, \ldots, z^q\}$ a collection of finitely many disjoint simple loops divided into two disjoint sets. Denote by $\bar{\Sigma}$ the surface obtained by collapsing the curves in Δ_n to points. Write

$$\Sigma^* := \bar{\Sigma} \setminus \Delta_p =: \coprod_{\nu=1}^{N} \Sigma^{(\nu)},$$

as a disjoint union of (not necessarily connected) components $\Sigma^{(\nu)}$. Let j be a conformal structure on $\Sigma \setminus \Delta$ such that $(\Sigma \setminus \Delta, j)$ is a punctured Riemann surface. This gives Σ^* the structure of a nodal punctured Riemann surface, with nodes corresponding to Δ_n and punctures corresponding to Δ_p. A *broken holomorphic curve (with N levels)*

$$F = (F^{(1)}, \ldots, F^{(N)}) : (\Sigma^*, j) \to (X^*, J^*)$$

is a collection of nodal punctured holomorphic curves $F^{(\nu)} : (\Sigma^{(\nu)}, j) \to (X^{(\nu)}, J^{(\nu)})$ such that $F : \Sigma^* \to X^*$ extends to a continuous map $\bar{F} : \bar{\Sigma} \to \bar{X}$. Moreover, we require that each level is *stable* in the following sense: For $1 \leq \nu \leq N$, $F^{(\nu)}$ is not the union of cylinders over closed Reeb orbits without any marked points on them. Moreover, Σ^* does not contain any sphere with less than three special points (punctures, nodal or marked points), nor a torus without special points, on which F is constant.

Note that, by continuity of $\bar{F}$, the number of positive punctures of $F^{(\nu)}$ agrees with the number of negative punctures of $F^{(\nu+1)}$, and the asymptotic Reeb orbits at the punctures agree correspondingly: $\overline{\Gamma}^{(\nu)} = \underline{\Gamma}^{(\nu+1)}$.

Remark 2.7 Every broken holomorphic curve has an underlying graph: Its vertices are the connected components of Σ^*, and each asymptotic Reeb orbit defines an edge between the corresponding components. Note that if Σ has genus zero the underlying graph is a tree.

We have the following easy

Lemma 2.8 ([19]) *The homology class $A := [\bar{F}] \in H_2(X; \mathbb{Z})$ of a nonconstant broken holomorphic curve $F : (\Sigma^*, j) \to (X^*, J^*)$ satisfies $\omega(A) > 0$.*

2.7 Gromov–Hofer compactness

For $R \in \mathbb{R}$ denote by

$$X_k \to X_0^- \cup_M [-k + R, R] \times M \cup_M X_0^+, \qquad x \mapsto x + R$$

the map which equals the identity on $X_0^- \cup X_0^+$ and is given by $(r, x) \mapsto (r + R, x)$ on $[-k, 0] \times M$. The following theorem collects the compactness properties that we need in this paper. A more precise statement is proved in [19].

Theorem 2.9 (Gromov–Hofer compactness) *Let (X_k, J_k) be as above and assume that (M, λ) is Morse–Bott. Let $f_k : (\Sigma_k, j_k) \to (X_k, J_k)$ be a sequence of holomorphic curves of the same genus and in the same homology class $[f_k] = A \in H_2(X; \mathbb{Z})$. After passing to a subsequence, there exist a broken holomorphic curve $F : (\Sigma^*, j) \to (X^*, J^*)$, orientation preserving diffeomorphisms $\phi_k : \Sigma_k \to \Sigma$, and numbers $-k = r_k^{(1)} < r_k^{(2)} < \cdots < r_k^{(N)} = 0$ with $r_k^{(\nu+1)} - r_k^{(\nu)} \to \infty$ such that the following holds:*

(i) $(\phi_k)_* j_k \to j$ *in* C^∞_{loc} *on* Σ^*.
(ii) $f_k^{(\nu)} \circ \phi_k^{-1} \to F^{(\nu)}$ *in* C^∞_{loc} *on* $\Sigma^{(\nu)}$, *where* $f_k^{(\nu)}$ *is the shifted map* $z \mapsto f_k(z) - r_k^{(\nu)}$.
(iii) $\int_{\Sigma_k} f_k^* \omega_k \to \int_{\Sigma^{(N)}} (F^{(N)})^* \omega^+$.
(iv) $[\bar{F}] = A \in H_2(X; \mathbb{Z})$.

Here on each component $\Sigma^{(\nu)}$ the convergence statements (i-ii) have to be understood in the sense of nodal holomorphic curves, see [19].

Corollary 2.10 *In the situation of Theorem 2.9, assume in addition that the genus is zero and the homology class A cannot be written as $A = B + C$ with $B, C \in H_2(X; \mathbb{Z})$ satisfying $\omega(B), \omega(C) > 0$. Then F is a broken holomorphic curve without nodes between nonconstant components and the convergence statements (i) and (ii) can be understood literally.*

Proof Suppose F has a node. Since the genus is zero, the node decomposes the domain Σ into two connected components Σ_0, Σ_1. The restrictions of F to these components define nonconstant broken holomorphic curves F_0, F_1 representing homology classes $A_0, A_1 \in H_2(X; \mathbb{Z})$ with $A_0 + A_1 = A$. Lemma 2.8 yields $\omega(A_0), \omega(A_1) > 0$, contradicting the assumption on A. □

2.8 Moduli spaces

Now we turn to moduli spaces of punctured holomorphic curves. From now on we restrict to genus zero. Consider a symplectic cobordism (X, ω) of dimension $2n$ with ends modelled over the contact manifolds $(\overline{M}, \overline{\lambda})$ and $(\underline{M}, \underline{\lambda})$. Suppose first that all closed Reeb orbits on $\overline{M}$ and $\underline{M}$ are nondegenerate. Fix collections of closed Reeb orbits

$$\overline{\Gamma} = (\overline{\gamma}_1, \ldots, \overline{\gamma}_{\overline{p}}), \qquad \underline{\Gamma} = (\underline{\gamma}_1, \ldots, \underline{\gamma}_{\underline{p}})$$

on $\overline{M}$ and $\underline{M}$, respectively. Fix also a relative homology class $A \in H_2(X, \overline{\Gamma} \cup \underline{\Gamma})$. Denote by $\widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma})$ the space of punctured holomorphic spheres asymptotic to $\overline{\Gamma}, \underline{\Gamma}$ in the homology class A. More precisely, elements of $\widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma})$ are triples $(f, \overline{z}, \underline{z})$, where $\overline{z} = (\overline{z}_1, \dots, \overline{z}_{\overline{p}})$ and $\underline{z} = (\underline{z}_1, \dots, \underline{z}_{\underline{p}})$ are collections of distinct points on the 2-sphere $S = S^2$, $\dot{S} := S^2 \backslash \{\overline{z}_i, \underline{z}_j\}$, and $f : \dot{S} \to X$ is a J-holomorphic map that is positively asymptotic to the closed Reeb orbit $\overline{\gamma}_i$ at the puncture $\overline{z}_i$, negatively asymptotic to $\underline{\gamma}_j$ at $\underline{z}_j$, and represents the relative homology class A.

The space $\widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma})$ is equipped with the topology induced by weighted Sobolev norms as in [11,13]. Denote its quotient by the action of the Möbius group by

$$\mathcal{M}^{A,J}(\overline{\Gamma}, \underline{\Gamma}) := \widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma}) \Big/ \mathrm{Aut}(S). \tag{3}$$

Its expected dimension is (see [13])

$$\begin{aligned} \dim \mathcal{M}^{A,J}(\overline{\Gamma}, \underline{\Gamma}) =& (n-3)(2 - \overline{p} - \underline{p}) + 2c_1(A) \\ &+ \sum_{i=1}^{\overline{p}} \mathrm{CZ}(\overline{\gamma}_i) - \sum_{j=1}^{\underline{p}} \mathrm{CZ}(\underline{\gamma}_j). \end{aligned}$$

Here the Conley–Zehnder indices are computed with respect to some fixed trivializations of the contact distribution ξ over $\overline{\gamma_i}$ and $\underline{\gamma_j}$. These trivializations induce trivializations of the tangent bundle TX over the asymptotic orbits, and $c_1(A)$ is the relative first Chern class of f^*TX with respect to these trivializations for any representative f of A.

If $\overline{M}$ and $\underline{M}$ are only Morse–Bott, then instead of collections of closed Reeb orbits we fix collections

$$\overline{\Gamma} = (\overline{\Gamma}_1, \dots, \overline{\Gamma}_{\overline{p}}), \qquad \underline{\Gamma} = (\underline{\Gamma}_1, \dots, \underline{\Gamma}_{\underline{p}})$$

of components of the orbifolds $\overline{\Gamma}_T$ and $\underline{\Gamma}_T$ of T-periodic orbits (for varying T). The preceding discussion carries over to this case, with the only difference that the expected dimension becomes (see [11, Corollary 5.4])

$$\begin{aligned} \dim \mathcal{M}^{A,J}(\overline{\Gamma}, \underline{\Gamma}) &= (n-3)(2 - \overline{p} - \underline{p}) + 2c_1(A) \\ &+ \sum_{i=1}^{\overline{p}} \Big(\mathrm{RS}(\overline{\Gamma}_i) + \frac{1}{2} \dim \overline{\Gamma}_i\Big) - \sum_{j=1}^{\underline{p}} \Big(\mathrm{RS}(\underline{\Gamma}_j) - \frac{1}{2} \dim \underline{\Gamma}_j\Big), \end{aligned}$$

or in view of Eq. (2),

$$\dim \mathcal{M}^{A,J}(\overline{\Gamma}, \underline{\Gamma}) = (n-3)(2-\overline{p}-\underline{p}) + 2c_1(A) + \sum_{i=1}^{\overline{p}} \big(\mathrm{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i\big) - \sum_{j=1}^{\underline{p}} \mathrm{CZ}(\underline{\Gamma}_j). \tag{4}$$

3 Tangency conditions

In this section we introduce holomorphic curves satisfying tangency conditions to a complex hypersurface, and we compute a specific invariant counting such curves in $\mathbb{CP}^n$. The discussion follows Section 6 in [20].

Consider a complex hypersurface in $Z = h^{-1}(0) \subset \mathbb{C}^n$ defined by a holomorphic function $h : \mathbb{C}^n \to \mathbb{C}$ with $h(0) = 0$ and $dh(0) \neq 0$. We say that a holomorphic map $f : \mathbb{C} \supset D \to \mathbb{C}^n$ with $f(0) = 0$ is *tangent of order* ℓ *to* Z *at* 0 if

$$(h \circ f)'(0) = \cdots = (h \circ f)^{(\ell)}(0) = 0.$$

This condition clearly only depends on the germs of f and Z at 0, and it is preserved under the action $(f, Z) \mapsto \big(\Phi \circ f \circ \phi^{-1}, \Phi(Z)\big)$ by local biholomorphisms of $(\mathbb{C}, 0)$ and $(\mathbb{C}^n, 0)$. Note that, after applying a local biholomorphism of $(\mathbb{C}^n, 0)$, we may assume that $h(z_1, \ldots, z_n) = z_n$ and the tangency condition becomes the condition $f'(0), \ldots, f^{(\ell)}(0) \in \mathbb{C}^{n-1}$ used in [20].

More generally, fix an integrable almost complex structure J_0 in a neighnorhood of a point $x \in X$ and the germ of a complex hypersurface Z through x. Consider an almost complex structure J which near x coincides with J_0 and a holomorphic map $f : \Sigma \to X$ from a Riemann surface with $f(z) = x$. We say that f is *tangent of order* ℓ *to* Z *at* z, and write $d^\ell f(z) \in T_x Z$, if $\Phi \circ f \circ \phi^{-1}$ is tangent of order ℓ to $\Phi(Z)$ at 0 for local holomorphic coordinates $\phi : (\Sigma, z) \to (\mathbb{C}, 0)$ and $\Phi : (X, x) \to (\mathbb{C}^n, 0)$.

Now suppose that X is a symplectic cobordism and fix collections $\overline{\Gamma}, \underline{\Gamma}$ of closed Reeb orbits on its ends. Denote by $\widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell)$ the space of punctured J-holomorphic spheres which are tangent of order ℓ to Z at x, i.e.,

$$\widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell) := \{(f, \overline{z}, \underline{z}, z) \mid (f, \overline{z}, \underline{z}) \in \widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma}), z \in \dot{S}, f(z) = x, d^\ell f(z) \in T_x Z\}.$$

Denote the quotient by the action of the Möbius group by

$$\mathcal{M}^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell) := \widetilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell) \Big/ \mathrm{Aut}(S).$$

The following result is a straighforward extension of [20, Proposition 6.9] to punctured holomorphic curves, with a slight modification concerning transversality of the evaluation map at z to the point x rather than Z.

Proposition 3.1 *For a generic almost complex structure J as above the moduli space $\mathcal{M}_s^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell) \subset \mathcal{M}^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell)$ of* simple *J-holomorphic spheres tangent of order ℓ to Z at x is a manifold of dimension*

$$\begin{aligned}\dim \mathcal{M}_s^{A,J}(\overline{\Gamma}, \underline{\Gamma}; x, Z, \ell) =&(n-3)(2-\overline{p}-\underline{p}) + 2c_1(A) + \sum_{i=1}^{\overline{p}} \mathrm{CZ}(\overline{\gamma}_i) \\ &- \sum_{j=1}^{\underline{p}} \mathrm{CZ}(\underline{\gamma}_j) - (2n-2) - 2\ell.\end{aligned}$$

3.1 Tangency conditions in cotangent bundles

We apply this proposition to the cotangent bundle T^*L of a manifold L^n which admits a metric of nonpositive curvature. For given $c > 0$, equip L with the Riemannian metric provided by Lemma 2.2 such that all closed geodesics of length $\leq c$ are noncontractible and nondegenerate and have Morse index $\leq n-1$. Let $\Gamma = (\gamma_1, \dots, \gamma_k)$ be a collection of (lifts of) closed geodesics of length $\leq c$. Fix a point $x \in T^*L$ and the germ of a complex hypersurface Z through x. Consider the moduli space $\mathcal{M}^J(\Gamma; x, Z, \ell)$ of holomorphic spheres with k positive punctures which are tangent of order ℓ to Z at x. For a punctured holomorphic sphere $f : \dot{S} \to T^*L$ trivialize the pullback bundle $f^*T(T^*L) \to \dot{S}$ and denote by $\mathrm{CZ}(\gamma_i)$ the Conley–Zehnder indices in this trivialization. Since the sum of the corresponding homology classes in L of the γ_i is null-homologous in L, the Maslov indices in this trivialization sum up to zero. Hence by Lemma 2.1, we can replace the Conley–Zehnder index by the Morse index in the dimension formula and obtain

$$\begin{aligned}\dim \mathcal{M}_s^J(\Gamma; x, Z, \ell) &= (n-3)(2-k) + \sum_{i=1}^{k} \mathrm{CZ}(\gamma_i) - (2n-2) - 2\ell \\ &\leq (n-3)(2-k) + k(n-1) - (2n-2) - 2\ell \\ &= 2k - 4 - 2\ell.\end{aligned}$$

Since by Proposition 3.1 this dimension is nonnegative for generic J, we obtain

Lemma 3.2 *For generic J every* simple *J-holomorphic sphere in T^*L which is asymptotic at the punctures to geodesics of length $\leq c$ and tangent of order ℓ to Z at x must have at least $\ell + 2$ punctures.*

Next we want to derive a version of this lemma for non-simple spheres. So consider $\tilde{f} = f \circ \phi$ for a simple sphere $f : \dot{S} \to T^*L$ as above and a d-fold branched covering $\phi : S^2 \to S^2$. Thus f has k positive punctures asymptotic to geodesics $\gamma_1, \dots, \gamma_k$ and $\tilde{f}$ has $\tilde{k} \geq k$ positive punctures asymptotic to multiples of the γ_i. We assume that $\tilde{f}$ is tangent of order $n-1$ to Z at a point $\tilde{p} \in S^2$. In suitable holomorphic coordinates near $\tilde{p}$ and $p = \phi(\tilde{p})$ we have $\phi(z) = z^b$, where $b = \mathrm{ord}(\tilde{p})$ is the branching order of $\tilde{p}$. In holomorphic coordinates near $x \in T^*L$ in which $Z = \mathbb{C}^{n-1} \subset \mathbb{C}^n$ we have $f_n(z) = a_1 z + a_2 z^2 + \cdots$ and

$$\tilde{f}_n(z) = a_1 z^b + a_2 z^{2b} + \cdots = O(z^n),$$

hence $a_1 = \cdots = a_\ell = 0$ where ℓ is the smallest integer $\geq n/b - 1$. By the Riemann-Hurwitz formula we have

$$2d - 2 = \sum_{z \in S^2} \big(\mathrm{ord}(z) - 1\big) = \sum_{i=1}^{\tilde{k}} \big(\mathrm{ord}(z_i) - 1\big) + \sum_{z \in \dot{S}} \big(\mathrm{ord}(z) - 1\big).$$

Since ϕ maps all punctures $\tilde{z}_i$ of $\tilde{f}$ to punctures of f, we have $\sum_{i=1}^{\tilde{k}} \mathrm{ord}(z_i) = kd$. The sum over $z \in \dot{S}$ is estimated below by the contribution $b - 1$ coming from $\tilde{p}$ and we obtain $2d - 2 \geq kd - \tilde{k} + b - 1$. Combining this with the estimates $k \geq \ell + 2$ from Lemma 3.2, $d \geq b$ and $\ell \geq n/b - 1$ we find

$$\tilde{k} \geq (k-2)d + b + 1 \geq (k-1)b + 1 \geq (\ell + 1)b + 1 \geq n + 1.$$

So we have shown

Corollary 3.3 *Let L^n be a manifold which admits a metric of nonpositive curvature. For $c > 0$, equip L with a metric such that all closed geodesics of length $\leq c$ are noncontractible and nondegenerate and have Morse index $\leq n-1$. Pick a point $x \in T^*L$ and the germ of a complex hypersurface Z through x. Then for generic J every (not necessarily simple) nonconstant J-holomorphic sphere in T^*L which is asymptotic at the punctures to geodesics of length $\leq c$ and tangent of order $n-1$ to Z at x must have at least $n+1$ punctures.*

3.2 Tangency conditions in $\mathbb{CP}^n$

The second situation we consider are non-punctured holomorphic spheres in $\mathbb{CP}^n$. Let $A = [\mathbb{CP}^1]$ and fix a point $x \in \mathbb{CP}^n$ and the germ of a local complex hypersurface Z through x (thus Z is only locally defined near x).

Since $c_1(A) = n + 1$, the dimension of the moduli space in Proposition 3.1 with $\ell = n - 1$ is

$$\dim \mathcal{M}^{A,J}(x, Z, \ell) = 2n - 6 + 2n + 2 - (2n - 2) - (2n - 2) = 0.$$

Note that all holomorphic spheres in class A are simple. By Gromov compactness, the zero-dimensional manifold $\mathcal{M}^{A,J}(x, Z, n-1)$ is compact. By the usual cobordism argument, the signed count of its points is independent of the regular almost complex structure J, the point x, and the complex hypersurface Z. Notice that no nodal curves occur in this count because holomorphic curves in the class A cannot split.

Proposition 3.4 *The signed count of points in $\mathcal{M}^{A,J}(x, Z, n-1)$ equals $(n-1)!$.*

By Gromov compactness, this implies

Corollary 3.5 *For every point $x \in \mathbb{CP}^n$, germ of complex hypersurface Σ through x, and compatibe almost complex structure J, there exists a J-holomorphic sphere in class $[\mathbb{CP}^1]$ which is tangent to order $n-1$ to Z at x.*

The proof of Proposition 3.4 is based on two lemmas.

Lemma 3.6 *For all positive real numbers $a_1, \ldots, a_n$ the system of n equations for n complex variables $z_1, \ldots, z_n$*

$$\begin{aligned} a_1 z_1 + \cdots + a_n z_n &= 0 \\ a_1 z_1^2 + \cdots + a_n z_n^2 &= 0 \\ \cdots& \\ a_1 z_1^n + \cdots + a_n z_n^n &= 0 \end{aligned} \tag{5}$$

has only the trivial solution $z_1 = \cdots = z_n = 0$.

Proof We prove the lemma by induction on n. The case $n = 1$ is clear. Suppose the lemma holds for $n - 1$ but not for n, so Eq. (5) has a nontrivial solution $z = (z_1, \ldots, z_n)$. Writing (5) as

$$\begin{pmatrix} a_1 & \ldots & a_n \\ a_1 z_1 & \ldots & a_n z_n \\ \ldots & \ldots & \ldots \\ a_1 z_1^{n-1} & \ldots & a_n z_n^{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_n \end{pmatrix} = 0,$$

this implies

$$\det\begin{pmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_n \\ \dots & \dots & \dots \\ z_1^{n-1} & \dots & z_n^{n-1} \end{pmatrix} = \prod_{i<j}(z_j - z_i) = 0.$$

So after reordering the z_i we may assume $z_{n-1} = z_n$. Inserting this in (5) and deleting the last equation, we obtain

$$\begin{aligned} &a_1 z_1 + \dots + (a_{n-1} + a_n) z_{n-1} = 0 \\ &a_1 z_1^2 + \dots + (a_{n-1} + a_n) z_{n-1}^2 = 0 \\ &\dots \\ &a_1 z_1^{n-1} + \dots + (a_{n-1} + a_n) z_{n-1}^{n-1} = 0. \end{aligned}$$

By hypothesis this has only the trivial solution $0 = z_1 = \dots = z_{n-1} = z_n$, contradicting the assumption $z \neq 0$. □

Lemma 3.7 *For all positive real numbers $a_0, \dots, a_n$ the n complex hypersurfaces in $\mathbb{CP}^n$ defined by the homogeneous equations*

$$\begin{aligned} &a_0 z_0 + \dots + a_n z_n = 0 \\ &a_0 z_0^2 + \dots + a_n z_n^2 = 0 \\ &\dots \\ &a_0 z_0^n + \dots + a_n z_n^n = 0 \end{aligned} \tag{6}$$

intersect transversally in $n!$ points.

Remark 3.8 It follows that the hypersurfaces defined by (6) intersect transversally for all complex numbers $a_0, \dots, a_n$ outside an algebraic subvariety in $\mathbb{C}^{n+1}$.

Proof Suppose $[z_0 : \dots : z_n]$ is a non-transverse intersection point. Then the matrix of linearized equations

$$\begin{pmatrix} a_0 & \dots & a_n \\ a_0 z_0 & \dots & a_n z_n \\ \dots & \dots & \dots \\ a_0 z_0^{n-1} & \dots & a_n z_n^{n-1} \end{pmatrix}$$

has rank $< n$, which implies

$$\det \begin{pmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{n-1} \\ \dots & \dots & \dots \\ z_0^{n-1} & \dots & z_{n-1}^{n-1} \end{pmatrix} = \prod_{i<j} (z_j - z_i) = 0.$$

So after reordering the z_i we may assume $z_0 = z_1$. Inserting this in (6), we obtain

$$\begin{aligned}
&(a_0 + a_1)z_1 + \cdots + a_n z_n = 0 \\
&(a_0 + a_1)z_1^2 + \cdots + a_n z_n^2 = 0 \\
&\dots \\
&(a_0 + a_1)z_0^n + \cdots + a_n z_n^n = 0.
\end{aligned}$$

By Lemma 3.6 this system has only the trivial solution $z_0 = z_1 = \cdots = z_n = 0$, contradicting the assumption $z \neq 0$.

By Bezout's theorem, the number of intersection points equals the product $1 \cdot 2 \cdots n$ of the degrees of the equations in (6). □

Proof of Proposition 3.4 Consider the standard complex structure on $\mathbb{CP}^n$. Denote by $\widetilde{\mathcal{M}}$ the space of holomorphic maps $f : \mathbb{CP}^1 \to \mathbb{CP}^n$ of degree 1 mapping $[1 : 0]$ to $[1 : 0 : \cdots : 0]$ and $[0 : 1]$ to the hyperplane $\{z_0 = 0\}$ at infinity. Each such map is of the form

$$f_p([z_0 : z_1]) = [z_0 : p_1 z_1 : \cdots : p_n z_1] \in \mathbb{CP}^n,$$

or in affine coordinates

$$f_p(z) = (p_1 z, \dots, p_n z) \in \mathbb{C}^n,$$

with $p = (p_1, \dots, p_n) \in \mathbb{C}^n \setminus 0$. The correspondence $p \mapsto f_p$ thus gives a diffeomorphism from $\mathbb{C}^n \setminus 0$ to $\widetilde{\mathcal{M}}$. Reparametrizing f as $f(\lambda z)$ for $\lambda \in \mathbb{C}^*$ corresponds to replacing p by λp, so the correspondence $p \mapsto f_p$ induces a diffeomorphism from $\mathbb{CP}^{n-1}$ to the moduli space $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{C}^*$ of degree 1 holomorphic spheres passing through 0 in $\mathbb{CP}^n$. It is well known (see [56, Proposition 7.4.3]) that this moduli space is transversely cut out and its orientation agrees with the complex orientation of $\mathbb{CP}^{n-1}$.

Let $Z \subset \mathbb{CP}^n$ be a complex hypersurface of degree $n - 1$ defined in affine coordinates $z = (z_1, \dots, z_n)$ by an equation

$$h(z) = h_1(z) + \cdots + h_{n-1}(z) = 0,$$

where $h_k(z) = a_1 z_1^k + \cdots + a_n z_n^k$ with positive real numbers $a_1, \ldots, a_n$. Note that $h_k \circ f_p(z) = z^k h_k(p)$ and thus $(h \circ f_p)^{(k)}(0) = k! h_k(p)$. So the jet evaluation map at 0

$$j : \widetilde{\mathcal{M}} \to \mathbb{C}^{n-1}, \qquad f_p \mapsto \Big((h \circ f_p)'(0), \ldots, (h \circ f_p)^{(n-1)}(0)\Big)$$

corresponds to the map

$$\mathbb{C}^n \backslash 0 \to \mathbb{C}^{n-1}, \qquad p \mapsto \Big(h_1(p), \ldots, (n-1)! h_{n-1}(p)\Big).$$

By Lemma 3.7 this map is transverse to $0 \in \mathbb{C}^{n-1}$ and its zero set in $\mathcal{M} \cong \mathbb{CP}^{n-1}$ consists of $(n-1)!$ points. Since all spaces are equipped with their complex orientations, all these points count positively and yield the signed count $(n-1)!$ of points in the space $\mathcal{M}^{A,J}(x, Z, n-1)$ in Proposition 3.4. □

4 Proofs requiring only standard transversality

In this section we prove all theorems from the Introduction except Theorem 1.2(b).

Proof of Theorem 1.1 Equip L with a Riemannian metric g_0 of nonpositive curvature such that the set $\{(q, p) \in T^*L \mid |p| \leq 2\}$ (with the symplectic form $dp\,dq$) embeds symplectically into $\mathbb{CP}^n$. Such a metric exists by the Lagrangian neighbourhood theorem and rescaling the metric. Perturb g_0 to a metric g as in Lemma 2.2 such that all closed g-geodesics of length $\leq \pi$ are noncontractible and nondegenerate and have Morse index $\leq n-1$. Clearly we can achieve that the unit ball cotangent bundle $W := \{(q, p) \in T^*L \mid |p| \leq 1\}$ with respect to g still embeds symplectically into $\mathbb{CP}^n$. Denote by $M := \partial W$ the unit cotangent bundle of L. We identify W and M with their images in $\mathbb{C}P^n$. Let J_M be an almost complex structure an $\mathbb{R} \times M$ compatible with the contact form $\lambda = p\,dq|_M$. Pick a compatible almost complex structure J on $\mathbb{CP}^n$ with $J = J_M$ near M. Let $(J_k)_{k \in \mathbb{N}}$ be the sequence of almost complex structures on $\mathbb{CP}^n$ obtained by the neck stretching procedure described before Theorem 2.9.

Fix a point x on L and the germ of a complex hypersurface $Z \subset T^*L$ through x. Choose J such that the conclusion of Corollary 3.3 holds (with $c = \pi$). By Corollary 3.5 there exists for each k a J_k-holomorphic sphere $f_k : S^2 \to \mathbb{CP}^n$ in the class $[\mathbb{CP}^1]$ which is tangent of order $n-1$ to Z at x. See Fig. 2.

Let $z_k \in S^2$ be such that

$$f_k(z_k) = x, \qquad d^{n-1} f_k(z_k) \in T_x Z.$$

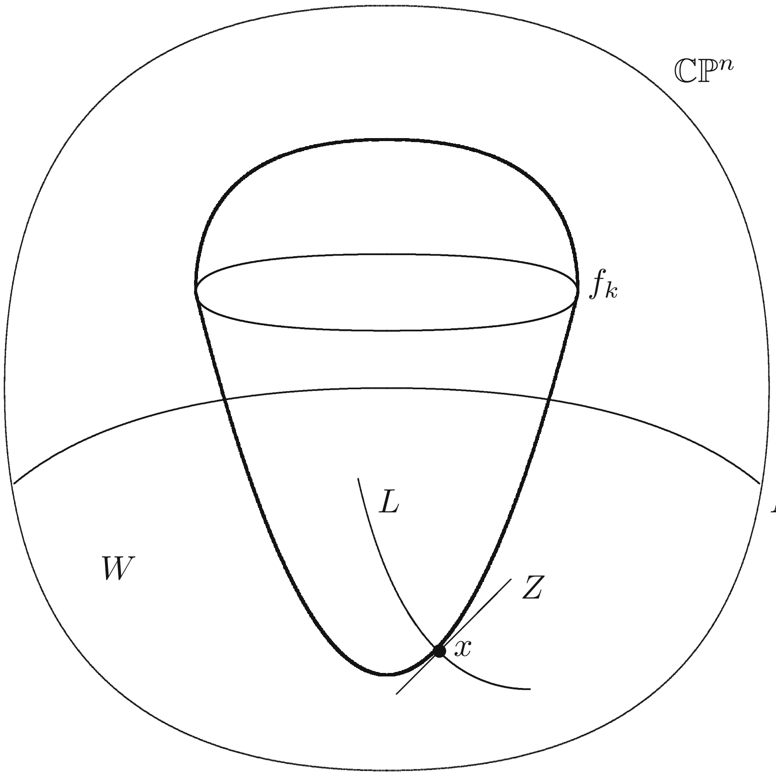

Fig. 2 A J_k-holomorphic sphere through $x \in L$ and tangent of order $(n-1)$ to Z

By Corollary 2.10, after passing to a subsequence, f_k converges to an N-level broken holomorphic curve $F = (F^{(1)}, \dots, F^{(N)}) : (\Sigma^*, j) \to (X^*, J^*)$ without nodes between nonconstant components. In particular, there exist diffeomorphisms $\phi_k : S^2 \to \Sigma$ such that $f_k^{(\nu)} \circ \phi_k^{-1}$ converges in C^∞_{loc} on $\Sigma^{(\nu)}$ to $F^{(\nu)} : \Sigma^{(\nu)} \to X^{(\nu)}$. Recall that $X^{(1)} = T^*L$, $X^{(N)} = \mathbb{C}P^n \backslash L$, and $X^{(\nu)} = \mathbb{R} \times M$ for $\nu = 2, \dots, N-1$.

Note that $x \in X_0^- \subset X^{(1)}$, and therefore $y_k := \phi_k(z_k) \in \Sigma^{(1)}$ with $f_k^{(1)} \circ \phi_k^{-1}(y_k) = x$ for all k. By the asymptotics of $F^{(1)}$ and the C^∞_{loc}-convergence $f_k^{(1)} \to F^{(1)}$, after passing to a subsequence we obtain $y_k \to y \in \Sigma^{(1)}$ with $F^{(1)}(y) = x$ and $d^{n-1}F^{(1)}(y) \in T_x Z$. Denote by C the component of $F^{(1)}$ containing y (see Fig. 3). Note that C is nonconstant because the limiting curve F has only one marked point and no nodes between nonconstant components.

Since Σ has genus zero, the graph underlying F is a tree (see the remark preceding Lemma 2.8). Let us replace each subtree emanating from the node C by *one* node which corresponds to a broken holomorphic curve with components in $\mathbb{C}\mathbb{P}^n \backslash L$, T^*L and $M \times \mathbb{R}$ and formally one negative puncture. (All other asymptotics appear in pairs of a negative and a positive puncture asymptotic to the same closed Reeb orbit). So we have a component C in T^*L with m positive punctures asymptotic to geodesics $\gamma_1, \dots, \gamma_m$ and m (broken) holomorphic planes C_i, $i = 1, \dots, m$, asymptotic to γ_i at their negative punc-

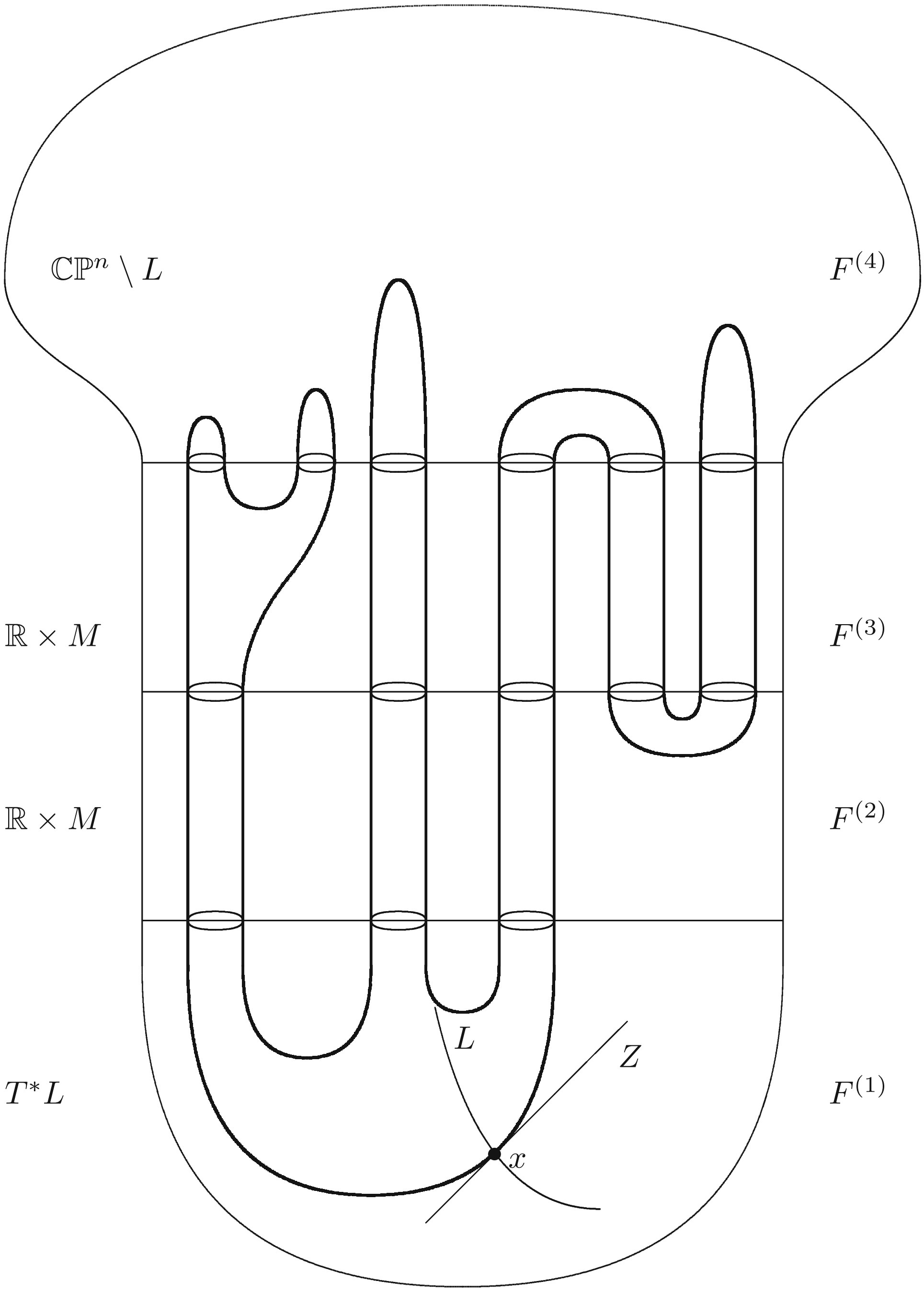

Fig. 3 A broken J_∞-holomorphic sphere through $x \in L$ and tangent of order $n-1$ to Z

tures. Since the γ_i are not contractible in L, each C_i must have a component in $\mathbb{C}P^n \setminus L$ and hence satisfies $\int_{C_i} \omega > 0$.

By Lemma 2.6 (applied to the broken holomorphic planes C_i), all the geodesics γ_i have length $\leq \pi$. So by Corollary 3.3, the number of punctures of C satisfies $m \geq n+1$. So the C_i give rise to $m \geq n+1$ nonconstant disks $g_i : D \to \mathbb{CP}^n$ with boundary on L. Since C_i is holomorphic, $g_i^*\omega \geq 0$ and $\int_D g_i^*\omega > 0$. Since $\sum_{i=1}^m \int_D g_i^*\omega = \pi$, the ω-area of one of the disks must be $\leq \pi/(n+1)$. $\square$

Proof of Theorem 1.2 (a) We retain the notation from the preceding proof. Assume that L is monotone. Then, since C_i has positive symplectic area, it must have positive Maslov index $\mu(C_i) \geq 1$. On the other hand, the sum of the Maslov indices equals the value of the first Chern class of $\mathbb{CP}^n$ on $[\mathbb{CP}^1]$,

$$\sum_{i=1}^{m} \mu(C_i) = 2n + 2.$$

Since $m \geq n + 1$, this implies that some $\mu(C_i) \leq 2$. If in addition L is orientable, then all Maslov indices are even, hence $m = n + 1$ and $\mu(C_i) = 2$ for all i. □

Proof of Theorem 1.16 We keep the notation of the proof of Theorem 1.1. Let W be a tubular neighbourhood of L as above with $W \cap B = \emptyset$ and let $M = \partial W$. Let J and the sequence J_k be as above. Moreover, choose J to be standard on the ball B. Let f_k be J_k-holomorphic spheres in the class of a complex line passing through the center p of B and a point x on L (they exist because the Gromov–Witten invariant of holomorphic spheres in the class of a complex line passing through two points in $\mathbb{C}P^n$ equals 1). In the limit $k \to \infty$ we find an N-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$ passing through x and p. Since L has no contractible geodesics, the component of $F^{(1)}$ passing through x must have at least 2 positive punctures. Hence $F^{(N)}$ has at least two components C_1, C_2. One of them, say C_1, passes through the point p. By the classical isoperimetric inequality (see [37]), C_1 has symplectic area at least πr^2. Therefore we have $0 < \int_{C_2} \omega \leq \pi - \pi r^2$. □

Proof of Theorem 1.17 Suppose L admits a Lagrangian embedding into a uniruled symplectic manifold (X, ω). Pick a tubular neighbourhood of L and almost complex structures J, J_k as in the proof of Theorem 1.1. Let f_k be J_k-holomorphic spheres passing through a point x on L (they exist by the definition of uniruledness). In the limit $k \to \infty$ we find an N-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$. Consider the simple curve underlying the component of $F^{(1)}$ which passes through x. It belongs to some moduli space of J^--holomorphic spheres in T^*L with m punctures passing through x. Since L has no contractible geodesics, we must have $m \geq 2$. Since all geodesics have Morse index zero, the moduli space is (if J^- is regular for simple curves passing through x) a manifold of dimension $(n-3)(2-m) - (2n-2)$. If $n \geq 3$ this dimension is negative and we have a contradiction. □

Proof of Theorem 1.21 Consider a Lagrangian embedding of L into a minimally uniruled symplectic manifold (X, ω). Construct an N-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$ of total area a passing through a point $x \in L$ as in the proof of Theorem 1.17. Since L has no contractible geodesics, the component of $F^{(1)}$ passing through x has at least 2 positive punctures. Hence $F^{(N)}$ has at least two components C_1, C_2. Since the total symplectic area of F equals a, both components must have area $0 < \int_{C_i} \omega < a$. So the Lagrangian embedding is not exact. □

Proof of Theorem 1.22 Consider $\alpha > 0$ such that the unit codisk bundle $D^*(Q, g)$ embeds symplectically into $(\mathbb{CP}^n, \alpha\omega)$. Pick an almost complex structure J as in Lemma 2.6. The proof of Theorem 1.1 yields a J-holomorphic plane $f : \mathbb{C} \to \mathbb{CP}^n \setminus L$ asymptotic to a closed geodesic γ whose symplectic area satisfies $\int_f \omega \leq \frac{\alpha\pi}{n+1}$. Since $\ell(\gamma) \leq \int_f \omega$ by Lemma 2.6, we conclude

$$\alpha \geq \frac{(n+1)\ell(\gamma)}{\pi} \geq \frac{(n+1)\ell_{\min}(Q, g)}{\pi}.$$

□

5 Proof of Theorem 1.2(b) assuming transversality

The remainder of the paper is devoted to the proof of Theorem 1.2(b) which requires stronger transversality. In this section we first explain the proof assuming the necessary transversality.

Proof of Theorem 1.2(b) Suppose that $L \subset \mathbb{CP}^n$ is a Lagrangian torus. We equip L with the standard flat metric, rescaled such that the unit ball cotangent bundle $W := \{(q, p) \in T^*L \mid |p| \leq 1\}$ embeds symplectically into $\mathbb{CP}^n$. Note that for this metric all closed geodesics occur in $(n-1)$-dimensional Morse–Bott families of Morse index zero.

Construct $C, C_1, \ldots, C_m$ as in the proof of Theorem 1.1. Thus C is a holomorphic sphere in T^*L with m positive punctures asymptotic to families of geodesics $\Gamma_1, \ldots, \Gamma_m$ and tangent of order $n-1$ to Z at $x \in L$, and C_i is a broken holomorphic plane negatively asymptotic to Γ_i. Assume that all components of C_i are regular and C is regular subject to the tangency condition to Z, and the evaluation maps at adjacent punctures are transverse to each other. It is shown in Sect. 9 that this can be achieved (by perturbing the Cauchy-Riemann equation with a domain dependent inhomogeneous term), so that C_i and C belong to moduli spaces $\mathcal{M}_i$ or $\mathcal{M}$, respectively, of dimensions

$$\dim \mathcal{M}_i = (n-3) - \mathrm{CZ}(\Gamma_i), \qquad i = 1, \ldots, m,$$

$$\dim \mathcal{M} = (n-3)(2-m) + (2n+2) + \sum_{1=1}^{m} \big(\mathrm{CZ}(\Gamma_i) + \dim \Gamma_i\big) - (4n-4).$$

Here the Conley–Zehnder indices are defined using trivializations over C_i or C, respectively. The $(2n+2)$ in the last line corresponds to the value of the first Chern class of $\mathbb{CP}^n$ on a complex line. The $-(4n-4)$ arises from $2n-2$ conditions for passing through x and $2n-2$ conditions for order $n-1$ tangency to Z. Note that the dimension formulae agree with the ones resulting from Eq. (4) for non-broken curves.

Denote by μ_i the Maslov index of C_i. Using $\dim \Gamma_i = n-1$ as well as $\sum_{i=1}^m \mu_i = 2n+2$ and the relation

$$\mathrm{CZ}(\Gamma_i) = \mathrm{ind}(\Gamma_i) - \mu_i = -\mu_i$$

from Lemma 2.1 we obtain

$$\begin{aligned}
\dim \mathcal{M}_i &= (n-3) + \mu_i, \qquad i = 1, \dots, m, \\
\dim \mathcal{M} &= (n-3)(2-m) + \sum_{\imath=1}^{m} \dim \Gamma_i - (4n-4) \\
&= (n-3)(2-m) + m(n-1) - (4n-4) \\
&= 2m - 2n - 2.
\end{aligned}$$

Note that $\dim \mathcal{M} \geq 0$ yields $m \geq n+1$. According to our regularity assumptions, the evaluation maps at the punctures

$$\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_m : \mathcal{M}_1 \times \cdots \times \mathcal{M}_m \to \Gamma := \Gamma_1 \times \cdots \times \Gamma_m, \qquad \mathrm{ev} : \mathcal{M} \to \Gamma$$

are transverse to each other. Hence the images of the linearized evaluation maps at any point must satisfy

$$\sum_{i=1}^{m} \dim \mathrm{im}(T\mathrm{ev}_i) + \dim \mathrm{im}(T\mathrm{ev}) \geq \dim \Gamma = m(n-1).$$

Inserting

$$\begin{aligned}
\dim \mathrm{im}(T\mathrm{ev}) &\leq \dim \mathcal{M} = 2m - 2n - 2, \\
\dim \mathrm{im}(T\mathrm{ev}_i) &\leq \min\{\dim \Gamma_i, \dim \mathcal{M}_i\} = \min\{n-1, n-3+\mu_i\} \\
&= \frac{1}{2}(2n - 4 + \mu_i - |2 - \mu_i|)
\end{aligned}$$

as well as $\sum \mu_i = 2n + 2$ we infer

$$\begin{aligned}
0 &\leq \sum_{i=1}^{m} \dim \operatorname{im}(T\mathrm{ev}_i) + \dim \operatorname{im}(T\mathrm{ev}) - \dim \Gamma \\
&\leq \frac{1}{2} \sum_{i=1}^{m} (2n - 4 + \mu_i - |2 - \mu_i|) + 2m - 2n - 2 - m(n-1) \\
&= \frac{1}{2} \sum_{i=1}^{m} (\mu_i - |2 - \mu_i|) + m - (2n + 2) \\
&= \frac{1}{2} \sum_{i=1}^{m} (-\mu_i - |2 - \mu_i|) + m,
\end{aligned}$$

hence

$$\sum_{i=1}^{m} (\mu_i + |2 - \mu_i|) \leq 2m.$$

Now note that $\mu_i + |2 - \mu_i| \geq 2$ for any number μ_i, with equality iff $\mu_i \leq 2$. It follows that $\mu_i \leq 2$ for all i. Since $\sum_{i=1}^{m} \mu_i = 2n + 2$ and all μ_i are even, at least $n + 1$ of the μ_i must be equal to 2. □

We see that the proof works provided that we can achieve regularity for all punctured holomorphic spheres (with suitable point and tangency constraints) in $T^*L, \mathbb{R} \times M$ and $X \setminus L$ resulting from the neck stretching procedure. There are three approaches towards proving such regularity: Kuranishi structures [33], polyfolds [43], and domain dependent perturbations [20]. In the remainder of this paper we will carry out the third approach.

6 Coherent perturbations

In this section we define a suitable class of perturbations of the Cauchy-Riemann equation parametrized by the Deligne–Mumford space $\overline{\mathcal{M}}_{k+1}$. The discussion closely follows [20], see also [36].

6.1 Nodal curves

Let us first describe the Deligne–Mumford space $\overline{\mathcal{M}}_k$ of stable curves of genus zero with k marked points. We adopt the approach and notation of [56, Appendix D], see also [20].

A *k-labelled tree* is a triple $T = (T, E, \Lambda)$, where (T, E) is a (connected) tree with set of vertices T and edge relation $E \subset T \times T$, and $\Lambda = \{\Lambda_\alpha\}_{\alpha \in T}$ is a decomposition of the index set $\{1, \ldots, k\} = \amalg_{\alpha \in T} \Lambda_\alpha$. We write $\alpha E \beta$ if $(\alpha, \beta) \in E$. Note that the labelling Λ defines a unique map $\{1, \ldots, k\} \to T$, $i \mapsto \alpha_i$ by the requirement $i \in \Lambda_{\alpha_i}$. Let

$$e(T) = |T| - 1$$

be the number of edges. A *tree homomorphism* $\tau : T \to \tilde{T}$ is a map which collapses some subtrees of T to vertices of $\tilde{T}$. A bijective tree homomorphism is called a *tree isomorphism.* A tree T is called *stable* if for each $\alpha \in T$,

$$n_\alpha := \#\Lambda_\alpha + \#\{\beta \mid \alpha E \beta\} \geq 3.$$

Note that for $k < 3$ every k-labelled tree is unstable. For $k \geq 3$, a k-labelled tree T can be canonically stabilized to a stable k-labelled tree $\mathrm{st}(T)$ by deleting the vertices with $n_\alpha < 3$ and modifying the edges in the obvious way.

A *nodal curve of genus zero with k marked points modelled over the tree* $T = (T, E, \Lambda)$ is a tuple

$$\mathbf{z} = (\{z_{\alpha\beta}\}_{\alpha E \beta}, \{z_i\}_{1 \leq i \leq k})$$

of points $z_{\alpha\beta}, z_i \in S^2$ such that for each $\alpha \in T$ the *special points*

$$SP_\alpha := \{z_{\alpha\beta} \mid \alpha E \beta\} \cup \{z_i \mid \alpha_i = \alpha\}$$

are pairwise distinct. Note that $n_\alpha = \#SP_\alpha$. For $\alpha \in T$ and $i \in \{1, \ldots, k\}$ denote by $z_{\alpha i}$ either the point z_i if $i \in \Lambda_\alpha$, or the point $z_{\alpha\beta_1}$ if $z_i \in \Lambda_{\beta_r}$ and $(\alpha, \beta_1), (\beta_1, \beta_2), \ldots, (\beta_{r-1}, \beta_r) \in E$. We associate to $\mathbf{z}$ the *nodal Riemann surface*

$$\Sigma_{\mathbf{z}} := \coprod_{\alpha \in T} S_\alpha \Big/ z_{\alpha\beta} \sim z_{\beta\alpha},$$

obtained by gluing a collection of standard spheres $\{S_\alpha\}_{\alpha \in T}$ at the points $z_{\alpha\beta}$ for $\alpha E \beta$, with marked points $z_i \in S_{\alpha_i}$, $i = 1, \ldots, k$. Note that $\mathbf{z}$ can be uniquely recovered from $\Sigma_{\mathbf{z}}$, so we will sometimes not distinguish between the two. A nodal curve $\mathbf{z}$ is called *stable* if the underlying tree is stable, i.e., every sphere S_α carries at least 3 special points. Stabilization of trees induces a canonical stabilization of nodal curves $\mathbf{z} \mapsto \mathrm{st}(\mathbf{z})$.

We will usually omit the genus zero from the notation. Denote the space of all nodal curves (of genus zero) with k marked points modelled over T by

$$\widetilde{\mathcal{M}}_T \subset (S^2)^E \times (S^2)^k.$$

Note that this is an open subset of the product of spheres. A *morphism* between nodal curves $\mathbf{z}$, $\tilde{\mathbf{z}}$ modelled over trees T, $\tilde{T}$ is a tuple

$$\phi = (\tau, \{\phi_\alpha\}_{\alpha \in T}),$$

where $\tau : T \to \tilde{T}$ is a tree homomorphism and $\phi_\alpha : S^2 \cong S_\alpha \to S_{\tau(\alpha)} \cong S^2$ are (possibly constant) holomorphic maps such that

$$\begin{aligned} \tilde{z}_{\tau(\alpha)\tau(\beta)} &= \phi_\alpha(z_{\alpha\beta}) \text{ if } \tau(\alpha) \neq \tau(\beta), \\ \phi_\alpha(z_{\alpha\beta}) &= \phi_\beta(z_{\beta\alpha}) \text{ if } \tau(\alpha) = \tau(\beta), \\ \tilde{\alpha}_i &= \tau(\alpha_i) \text{ and } \qquad \tilde{z}_i = \phi_{\alpha_i}(z_i) \end{aligned}$$

for $i = 1, \dots, k$ and $\alpha E \beta$. A morphism $\phi : \mathbf{z} \to \tilde{\mathbf{z}}$ induces a natural holomorphic map $\Sigma_{\mathbf{z}} \to \Sigma_{\tilde{\mathbf{z}}}$ (i.e., a continuous map that is holomorphic on each component S_α). A morphism $(\tau, \{\phi_\alpha\})$ is called *isomorphism* if τ is a tree isomorphism and each ϕ_α is biholomorphic.

Isomorphisms from $\mathbf{z}$ to itself are called *automorphisms*. If $\mathbf{z}$ is stable its only automorphism is the identity (see [56], discussion after Definition D.3.4). Thus for a stable tree T we have a free and proper holomorphic action

$$G_T \times \widetilde{\mathcal{M}}_T \to \widetilde{\mathcal{M}}_T$$

of the group G_T of isomorphisms fixing T, i.e., such that $\tau = \mathrm{id} : T \to T$ in the definition of a morphism and thus $G_T \cong \prod_{\alpha \in T} \mathrm{Aut}(S^2)$. Hence the quotient

$$\mathcal{M}_T := \widetilde{\mathcal{M}}_T / G_T$$

is a complex manifold of dimension

$$\dim_{\mathbb{C}} \mathcal{M}_T = k + 2e(T) - 3|T| = k - 3 - e(T).$$

For $k \geq 3$, denote by $\mathcal{M}_k = \widetilde{\mathcal{M}}_k / G$ the moduli space of stable curves modelled over the k-labelled tree with one vertex. As a set, the *Deligne–Mumford space (of genus zero) with k marked points* is given by

$$\overline{\mathcal{M}}_k := \coprod_T \mathcal{M}_T,$$

where the union is taken over the (finitely many) isomorphism classes of stable k-labelled trees. However, $\overline{\mathcal{M}}_k$ is equipped with the topology of Gromov convergence which makes it a compact connected metrizable space (see [56]).

As the notation suggests, $\overline{\mathcal{M}}_k$ is the compactification of $\mathcal{M}_k$ in the Gromov topology. For a stable k-labelled tree T, the closure of $\mathcal{M}_T$ in $\overline{\mathcal{M}}_k$ is given by

$$\overline{\mathcal{M}}_T = \coprod_{\tilde{T}} \mathcal{M}_{\tilde{T}},$$

where the union is taken over all isomorphism classes of stable k-labelled trees $\tilde{T}$ for which there exists a surjective tree homomorphism $\tau : \tilde{T} \to T$ with $\tau(\tilde{\alpha}_i) = \alpha_i$ for $i = 1, \ldots, k$.

We have the following result of Knudsen (cf. [56]):

Theorem 6.1 *For $k \geq 3$, the Deligne–Mumford space $\overline{\mathcal{M}}_k$ is a compact complex manifold of dimension* $\dim_{\mathbb{C}} \overline{\mathcal{M}}_k = k - 3$*. Moreover, for each stable k-labelled tree T, the space $\overline{\mathcal{M}}_T \subset \overline{\mathcal{M}}_k$ is a compact complex submanifold of codimension* $\operatorname{codim}_{\mathbb{C}} \overline{\mathcal{M}}_T = e(T)$.

6.2 Stable decompositions

Consider the Deligne–Mumford space $\overline{\mathcal{M}}_{k+1}$ with $k + 1$ marked points $z_0, \ldots, z_k$. In the following discussion, the point z_0 plays a special role (it will be the variable for holomorphic maps in later sections).

We have a canonical projection $\pi : \overline{\mathcal{M}}_{k+1} \to \overline{\mathcal{M}}_k$ by forgetting the marked point z_0 and stabilizing. The map π is holomorphic and the fibre $\pi^{-1}([\mathbf{z}])$ is naturally biholomorphic to $\Sigma_{\mathbf{z}}$. The projection

$$\pi : \overline{\mathcal{M}}_{k+1} \to \overline{\mathcal{M}}_k$$

is called the *universal curve*.

Given a stable $(k + 1)$-labelled tree T, we define an equivalence relation on $\{0, \ldots, k\}$ by $i \sim j$ iff $z_{\alpha_0 i} = z_{\alpha_0 j}$ (note that this depends only on the labelled tree and not the actual positions of the points on the spheres). The equivalence classes yield a decomposition

$$\{0, \ldots, k\} = I_0 \cup \cdots \cup I_\ell.$$

Note that the marked points on S_{α_0} correspond to equivalence classes consisting of one element; in particular, we may put $I_0 := \{0\}$. Stability implies $\ell + 1 = n_{\alpha_0} \geq 3$. Conversely, we call a decomposition $\mathbf{I} = (I_0, \ldots, I_\ell)$ of $\{0, \ldots, k\}$ *stable* if $I_0 = \{0\}$ and $|\mathbf{I}| := \ell + 1 \geq 3$. We will always order the I_j such that the integers $i_j := \min\{i \mid i \in I_j\}$ satisfy

$$0 = i_0 < i_1 < \cdots < i_\ell.$$

Denote by $\mathcal{M}_{\mathbf{I}} = \mathcal{M}_{(I_0,\dots,I_\ell)} \subset \overline{\mathcal{M}}_{k+1}$ the union over those stable trees that give rise to the stable decomposition $\mathbf{I} = (I_0, \dots, I_\ell)$. The $\mathcal{M}_{\mathbf{I}}$ are submanifolds of $\overline{\mathcal{M}}_{k+1}$ with

$$\overline{\mathcal{M}}_{k+1} = \bigcup_{\mathbf{I}} \mathcal{M}_{\mathbf{I}}.$$

The closure $\overline{\mathcal{M}}_{\mathbf{J}}$ is a closed complex submanifold of $\overline{\mathcal{M}}_{k+1}$ which is a union of certain strata $\mathcal{M}_{\mathbf{I}}$ with $|\mathbf{I}| \le |\mathbf{J}|$. The above ordering of the I_j determines a projection $p_{\mathbf{I}} : \mathcal{M}_{\mathbf{I}} \to \mathcal{M}_{|\mathbf{I}|}$, sending a stable curve $\mathbf{z}$ to the special points on the component S_{α_0}. It extends to a smooth map between the closures which we denote by

$$p_{\mathbf{I}} : \overline{\mathcal{M}}_{\mathbf{I}} \to \overline{\mathcal{M}}_{|\mathbf{I}|}.$$

We call the union of the strata $\mathcal{M}_{\mathbf{I}}$ with $|\mathbf{I}| = 3$ the *set of special points* of $\overline{\mathcal{M}}_{k+1}$. On its complement $\overline{\mathcal{M}}^*_{k+1}$, the projection

$$\pi : \overline{\mathcal{M}}^*_{k+1} \to \overline{\mathcal{M}}_k$$

is a fibration. We denote by

$$T^v\overline{\mathcal{M}}^*_{k+1} := \ker(d\pi) \to \overline{\mathcal{M}}^*_{k+1}$$

the *vertical tangent bundle*.

6.3 Coherent perturbations

Fix a number δ with $0 < \delta < \operatorname{arsinh}(1)$ and consider a complete hyperbolic surface (Σ, h), i.e., a surface Σ equipped with a complete metric h of curvature -1 and finite volume. The *δ-thick part* Σ^δ of (Σ, h) is defined as the set of all points at which the injectivity radius is greater or equal to δ. For $k \ge 3$ we denote by $\overline{\mathcal{M}}^\delta_{k+1} \subset \overline{\mathcal{M}}^*_{k+1}$ the union of the δ-thick parts of all fibres of $\pi : \overline{\mathcal{M}}^*_{k+1} \to \overline{\mathcal{M}}_k$ with respect to their unique complete hyperbolic metrics. This is a compact submanifold with boundary, and the condition $\delta < \operatorname{arsinh}(1)$ ensures that its interior meets each component of each fibre, see [45] or Appendix D.

Now let $(E, J) \to X$ be a complex vector bundle over a compact manifold X (possibly with boundary). The pullbacks under the obvious projections give complex vector bundles $(p_1^* T^v\overline{\mathcal{M}}^*_{k+1}, j)$ and $(p_2^* E, J)$ over the product $\overline{\mathcal{M}}^*_{k+1} \times X$. We denote by

$$\mathrm{Hom}^{0,1}(p_1^*T^v\overline{\mathcal{M}}^*_{k+1}, p_2^*E) \to \overline{\mathcal{M}}^*_{k+1} \times X \tag{7}$$

the bundle of complex antilinear bundle homomorphisms. Note that for every stable decomposition $\mathbf{I}$ with $|\mathbf{I}| \geq 4$, the restriction of this bundle to $\overline{\mathcal{M}}^*_{\mathbf{I}} \times X$ where $\overline{\mathcal{M}}^*_{\mathbf{I}} := \overline{\mathcal{M}}_{\mathbf{I}} \cap \overline{\mathcal{M}}^*_{k+1}$ is the pullback bundle

$$\begin{array}{ccc} \mathrm{Hom}^{0,1}(p_1^*T^v\overline{\mathcal{M}}_{\mathbf{I}}, p_2^*E) & \longrightarrow & \mathrm{Hom}^{0,1}(p_1^*T^v\overline{\mathcal{M}}_{|\mathbf{I}|}, p_2^*E) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}^*_{\mathbf{I}} \times X & \xrightarrow{p_{\mathbf{I}} \times \mathrm{id}} & \overline{\mathcal{M}}^*_{|\mathbf{I}|} \times X \end{array} \tag{8}$$

Definition 6.2 A *coherent perturbation* on (E, J) is a continuous section K of the bundle (7) satisfying the following two conditions:

(a) K has support in the compact set $\overline{\mathcal{M}}^\delta_{k+1} \times X$;
(b) for every stable decomposition $\mathbf{I}$ with $|\mathbf{I}| \geq 4$, the restriction $K|_{\overline{\mathcal{M}}_{\mathbf{I}} \times X}$ is the pullback under $p_{\mathbf{I}} \times \mathrm{id}$ of a smooth section of the right-hand bundle in (8).

In view of condition (a), we may view K as being extended by zero to all of $\overline{\mathcal{M}}_{k+1}$ (although the bundle (7) is not defined at the special points). Note that the definition depends on the choice of δ, but we will not indicate this dependence in the notation. We denote the space of coherent perturbations on (E, J) by

$$\mathcal{K}(\overline{\mathcal{M}}_{k+1}, E).$$

It is equipped with the C^0-topology on $\overline{\mathcal{M}}_{k+1}$ and the C^∞-topology on each $\mathcal{M}_{\mathbf{I}}$ via the pullback diagram (8).

To better understand Definition 6.2, fix a stable curve $\mathbf{z} \in \overline{\mathcal{M}}_k$, modelled over the k-labelled tree T. Recall that $\pi^{-1}[\mathbf{z}]$ is naturally identified with the nodal Riemann surface $\Sigma_{\mathbf{z}}$ which is a union of $|T|$ copies S_α of S^2, glued together at the points $z_{\alpha\beta}$. Denote by $\Sigma^*_{\mathbf{z}} \subset \Sigma_{\mathbf{z}}$ the complement of the special points, which is a smooth (possibly disconnected) punctured Riemann surface. Restriction of $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, E)$ to $\pi^{-1}[\mathbf{z}]$ yields a smooth section $K_{\mathbf{z}}$ with compact support in the bundle

$$\mathrm{Hom}^{0,1}(p_1^*T\Sigma^*_{\mathbf{z}}, p_2^*E) \to \Sigma^*_{\mathbf{z}} \times X. \tag{9}$$

In other words, $K_{\mathbf{z}}$ is a compactly supported $(0, 1)$-form on $\Sigma^*_{\mathbf{z}}$ with values in the sections of E. If $E = TX$ is the tangent bundle of an almost complex manifold (X, J), then $K_{\mathbf{z}}$ is a $(0, 1)$-form on $\Sigma^*_{\mathbf{z}}$ with values in the vector fields on X.

In general, for any (possibly disconnected) complete hyperbolic surface Σ, we denote by

$$\mathcal{K}(\Sigma, E) \tag{10}$$

the space of smooth sections in the bundle

$$\mathrm{Hom}^{0,1}(p_1^* T\Sigma, p_2^* E) \to \Sigma \times X \tag{11}$$

with support in the δ-thick part Σ^δ. The proof of the following lemma is analogous to the proof of [20, Lemma 3.10].

Lemma 6.3 *For $\mathbf{z} \in \overline{\mathcal{M}}_k$ and $K_\mathbf{z} \in \mathcal{K}(\Sigma_\mathbf{z}^*, E)$, there exists a $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, E)$ whose restriction to $\pi^{-1}[\mathbf{z}]$ equals $K_\mathbf{z}$.* □

6.4 K-holomorphic maps

Let (X, J) be an almost complex manifold, Σ a Riemann surface, and $K \in \mathcal{K}(\Sigma, TX)$. Then to a map $f : \Sigma \to X$ we associate the smooth section

$$\bar{\partial}_{J,K} f := df + J(f) \circ df \circ j + K(f)$$

of the bundle $\mathrm{Hom}^{0,1}(T\Sigma, f^*TX) \to \Sigma$ whose value at $z \in \Sigma$ is the complex antilinear homomorphism

$$d_z f + J\big(f(z)\big) \circ d_z f \circ j_z + K(z, f(z)) \ : \ T_z\Sigma \to T_{f(z)}X.$$

A map f satisfying $\bar{\partial}_{J,K} f = 0$ will be called *K-holomorphic*.

More globally, a coherent perturbation $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX)$ associates to each $\mathbf{z} \in \overline{\mathcal{M}}_k$ a coherent perturbation $K_\mathbf{z} \in \mathcal{K}(\Sigma_\mathbf{z}^*, TX)$, and thus to a map $f : \Sigma_\mathbf{z}^* \to X$ the smooth section $\bar{\partial}_{J,K_\mathbf{z}} f$ of the bundle $\mathrm{Hom}^{0,1}(T\Sigma_\mathbf{z}^*, f^*TX) \to \Sigma_\mathbf{z}^*$.

6.5 Taming conditions

Let (X, J) be an almost complex manifold, Σ a Riemann surface, and $K \in \mathcal{K}(\Sigma, TX)$. Then a K-holomorphic map $\Sigma \to X$ can be viewed as a holomorphic section for the almost complex structure $\widehat{J}$ on $\Sigma \times X$ defined by the matrix

$$\widehat{J} := \begin{pmatrix} j & 0 \\ -K \circ j & J \end{pmatrix}$$

with respect to the splitting $T(\Sigma \times X) = T\Sigma \oplus TX$. Here $\widehat{J}^2 = -\mathbb{1}$ follows from $K \circ j = -J \circ K : T\Sigma \to TX$, and a short computation shows that

$f : \Sigma \to X$ is K-holomorphic iff $\widehat{f} := \mathrm{id} \times f : \Sigma \to \Sigma \times X$ is $\widehat{J}$-holomorphic. Another short computation shows that the projection $\Sigma \times X \to \Sigma$ is holomorphic.

For an area form σ on Σ and a compatible symplectic form ω on X, we obtain a symplectic form $\widehat{\omega} := \sigma \oplus \omega$ on $\Sigma \times X$. The following computation shows that $\widehat{\omega}$ tames $\widehat{J}$ if the C^0-norm of K, measured with respect to the metrics $\sigma(\cdot, j\cdot)$ and $\omega(\cdot, J\cdot)$, satisfies $\|K\| \le 1$:

$$\begin{aligned}
\widehat{\omega}\left(\begin{pmatrix} v \\ w \end{pmatrix}, \widehat{J}\begin{pmatrix} v \\ w \end{pmatrix}\right) &= \widehat{\omega}\left(\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} jv \\ Jw - Kjv \end{pmatrix}\right) \\
&= \sigma(v, jv) + \omega(w, Jw) - \omega(w, Kjv) \\
&\ge |v|^2 + |w|^2 - \|K\|\,|v|\,|w| \\
&\ge \frac{1}{2}(|v|^2 + |w|^2).
\end{aligned}$$

Next, consider an oriented manifold M^{2n-1} with a *stable Hamiltonian structure* (ω, λ), i.e., a closed 2-form ω and a 1-form λ satisfying

$$\ker(\omega) \subset \ker(d\lambda), \qquad \lambda \wedge \omega^{n-1} > 0.$$

It determines a *Reeb vector field* R by the conditions $i_R\lambda = 1$ and $i_R\omega = 0$. Note that a contact form λ induces a stable Hamiltonian structure $(d\lambda, \lambda)$. As in the contact case, an almost complex structure J on $\mathbb{R} \times M$ is called *compatible with* (ω, λ) if J is $\mathbb{R}$-invariant, preserves $\xi = \ker\lambda$, maps ∂_r to the Reeb vector field R, and $J|_\xi$ is compatible with ω.

For a complete hyperbolic surface (Σ, h) with area form σ, the pair

$$(\widehat{\omega} := \sigma \oplus \omega, \widehat{\lambda} := \lambda) \tag{12}$$

defines a stable Hamiltonian structure on $\Sigma \times M$. Let J be an almost complex structure on $\mathbb{R} \times M$ compatible with (ω, λ) and consider the space

$$\mathcal{K}(\Sigma, \xi)$$

of smooth sections in the bundle $\mathrm{Hom}^{0,1}(p_1^*T\Sigma, p_2^*\xi) \to \Sigma \times M$ with support in the δ-thick part Σ^δ. Note that $K \in \mathcal{K}(\Sigma, \xi)$ canonically gives rise to a coherent perturbation on the bundle $T(\mathbb{R} \times M) \to M$ whose components in $\mathbb{R} \oplus \mathbb{R} \cdot R$ vanish identically. For $K \in \mathcal{K}(\Sigma, \xi)$ we define an $\mathbb{R}$-invariant almost complex structure $\widehat{J}$ on $\Sigma \times (\mathbb{R} \times M)$ by $\widehat{J}(\partial_r) = R$ and the matrix

$$\widehat{J}|_{\widehat{\xi}} := \begin{pmatrix} j & 0 \\ -K \circ j & J|_\xi \end{pmatrix} \tag{13}$$

on $\widehat{\xi} := \ker \widehat{\lambda} = T\Sigma \oplus \xi$. Note that $\widehat{J}|_{\widehat{\xi}}$ is tamed by $\widehat{\omega}$ provided that $\|K\| \leq 1$.

6.6 Globalization and compactness

To globalize this construction, we fix a Kähler form σ on the Deligne–Mumford space $\overline{\mathcal{M}}_{k+1}$. Such a Kähler form can, for example, be obtained from the embedding of $\overline{\mathcal{M}}_{k+1}$ into a product of $\mathbb{CP}^1$'s constructed in [56]. Now every component S_α of a fibre $\pi^{-1}(\mathbf{z})$ of the projection $\pi : \overline{\mathcal{M}}_{k+1} \to \overline{\mathcal{M}}_k$ is an embedded holomorphic sphere, so $\sigma|_{S_\alpha}$ defines a positive area form on S_α. For a compact almost complex manifold (X, J) with compatible symplectic form ω and a coherent perturbation $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX)$, the construction above yields a complex structure $\widehat{J}$ on the vertical tangent bundle (outside the special points) of the projection $\overline{\mathcal{M}}_{k+1} \times X \to \overline{\mathcal{M}}_k$. This complex structure is tamed by the restriction of $\widehat{\omega} = \sigma \oplus \omega$ to the fibres provided that the C^0-norm of K, measured with respect to the metrics $\sigma(\cdot, j\cdot)$ and $\omega(\cdot, J\cdot)$, satisfies $\|K\| \leq 1$.

Next, we define coherent perturbations on the noncompact symplectic manifolds considered in this paper. Here all manifolds are assumed to be equipped with a compatible almost complex structure J.

Definition 6.4 (i) For the symplectization $\mathbb{R} \times M$ of a closed manifold M with stable Hamiltonian structure (ω, λ) we consider the space

$$\mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$$

of coherent perturbations on the bundle $\xi = \ker \lambda \to M$. We canonically extend each $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$ to the bundle $T(\mathbb{R} \times M) \to \mathbb{R} \times M$, $\mathbb{R}$-invariantly on the base and vanishing on the subbundle $\mathbb{R} \oplus \mathbb{R} \cdot R$.

(ii) For a symplectic cobordism $X = X_0 \cup (\mathbb{R}_+ \times \overline{M}) \cup (\mathbb{R}_- \times \underline{M})$ we define

$$\mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX) \subset \mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX_0)$$

to consist of those K which agree to infinite order with elements of $\mathcal{K}(\overline{\mathcal{M}}_{k+1}, \overline{\xi})$ and $\mathcal{K}(\overline{\mathcal{M}}_{k+1}, \underline{\xi})$ on $\overline{M}$ and $\underline{M}$, respectively (hence they extend $\mathbb{R}$-invariantly to the ends of X).

(iii) For a split symplectic cobordism $X^* = X^+ \amalg (\mathbb{R} \times M) \amalg X^-$ we define

$$\mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX^*) \subset \mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX_0)$$

to consist of those K on the compact manifold $X_0 = X_0^- \cup_M X_0^+$ which agree to infinite order with a element of $\mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$ on M (hence they extend $\mathbb{R}$-invariantly to $\mathbb{R} \times M$ and the ends of $X^\pm$).

The Compactness Theorem 2.9 continues to hold for K-holomorphic maps. We refer to Appendix D for the precise statement and its proof.

Remark 6.5 In the preceding discussion we have used the setting of stable Hamiltonian structures because this is most natural for the graph construction. If the cylindrical ends are modelled over contact manifolds (which is the case we are interested in), one could avoid the use of stable Hamiltonian structures as follows. Consider a contact manifold (M, λ). For a fixed open Riemann surface Σ, one considers a 1-form β on Σ such that $d\beta = \sigma$ is an area form and a family of contact forms λ_z on M depending with compact support on $z \in \Sigma$. Then $\widehat{\lambda} = \lambda_\bullet + \beta$ defines a contact form on $\Sigma \times M$ provided the derivatives of λ_z with respect to z are sufficiently small. Now one considers almost complex structures $\widehat{J}$ on $\Sigma \times (\mathbb{R} \times M)$ compatible with (or tamed by) $\widehat{\lambda}$. To globalize the construction, one picks a primitive β of the restriction of the Kähler form σ on $\overline{\mathcal{M}}_{k+1}$ to the open stratum $\mathcal{M}_{k+1}$ (which exists because the codimension two homology of $\overline{\mathcal{M}}_{k+1}$ is generated by boundary divisors). The corresponding holomorphic curves will again satisfy Gromov-Hofer compactness. However, the computations in the following section would be more involved for this class of perturbations.

6.7 A genericity result

Consider the symplectization $\mathbb{R} \times M$ of a closed manifold M with stable Hamiltonian structure (ω, λ), a coherent perturbation $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$, and a point $\mathbf{z} \in \overline{\mathcal{M}}_k$. Then a map $f = (a, u) : \Sigma_{\mathbf{z}}^* \to \mathbb{R} \times M$ is $K_{\mathbf{z}}$-holomorphic iff it satisfies the equations

$$da - u^*\lambda \circ j = 0, \qquad \pi_\xi du + J(u) \circ \pi_\xi du \circ j + K(z, u) = 0, \tag{14}$$

where $\pi_\xi : TM \to \xi$ is the projection along the Reeb vector field R. The second equation implies that $\pi_\xi du(z) \neq 0$ at points where $K(z, u(z)) \neq 0$, in particular f cannot be a nontrivial branched cover of an orbit cylinder unless $K(z, u(z))$ vanishes identically. The following lemma refines this observation. Recall that $\Sigma_{\mathbf{z}}^\delta$ denotes the δ-thick part of the hyperbolic surface $\Sigma_{\mathbf{z}}^*$. Also recall that a subset of a topological space is called *Baire subset* if it contains a countable intersection of open dense sets.

Lemma 6.6 *Suppose that* $\dim M \geq 3$. *Then there exists a Baire subset* $\mathcal{K}^*(\overline{\mathcal{M}}_{k+1}, \xi) \subset \mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$ *such that for every* $K \in \mathcal{K}^*(\overline{\mathcal{M}}_{k+1}, \xi)$, $\mathbf{z} \in \overline{\mathcal{M}}_k$, $U \subset \Sigma_{\mathbf{z}}^\delta$ *open nonempty, and every* $K_{\mathbf{z}}$*-holomorphic map* $f = (a, u) : U \to \mathbb{R} \times M$ *the set* $\{z \in U \mid \pi_\xi du(z) \neq 0\}$ *is open and dense in* U.

Proof Recall that elements of $\mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$ are continuous sections of the vector bundle

$$\mathrm{Hom}^{0,1}(p_1^* T^v \overline{\mathcal{M}}_{k+1}^*, p_2^* \xi) \to \overline{\mathcal{M}}_{k+1}^* \times M$$

with support in the δ-thick part $\overline{\mathcal{M}}_{k+1}^\delta \times M$ such that for every stable decomposition $\mathbf{I}$ with $|\mathbf{I}| \geq 4$, the restriction $K|_{\overline{\mathcal{M}}_{\mathbf{I}}^* \times M}$ is the pullback of a smooth section of the bundle

$$\mathrm{Hom}^{0,1}(p_1^* T^v \overline{\mathcal{M}}_{|\mathbf{I}|}^*, p_2^* \xi) \to \overline{\mathcal{M}}_{|\mathbf{I}|}^* \times M.$$

Denote this bundle by $E_{\mathbf{I}} \to B_{\mathbf{I}}$ and note that the interior of $\overline{\mathcal{M}}_{|\mathbf{I}|}^\delta \times M$ defines an open subset $B_{\mathbf{I},0} \subset B_{\mathbf{I}}$ with compact closure. If $\dim M = 2n - 1$ with $n \geq 2$, then the real rank of $E_{\mathbf{I}}$ is $rk(E_{\mathbf{I}}) = (2n-2) \geq 2$. Let $\mathcal{F}_{\mathbf{I}}$ be the 3-dimensional foliation of $B_{\mathbf{I}}$ with leaves $\Sigma_{\mathbf{z}}^* \times \gamma$ for $\mathbf{z} \in \pi(\overline{\mathcal{M}}_{|\mathbf{I}|}^*)$ and (not necessarily closed) Reeb orbits $\gamma \subset M$. Then we are in the setting of Proposition E.1 in Appendix E with $q = (2n-2)$, $m = 3$ and $k := 2$, so $k > m - q$ because $n \geq 2$. Denote by $\mathcal{K}_{\mathbf{I}} = \Gamma(E_I; B_{\mathbf{I},0}, 0)$ the space of smooth sections in the bundle $E_{\mathbf{I}}$ with support in the closure of $B_{\mathbf{I},0}$, and let $\mathcal{K}_{\mathbf{I}}^* \subset \mathcal{K}_{\mathbf{I}}$ be the Baire subset provided by Proposition E.1.

Let $\mathcal{K}^*(\overline{\mathcal{M}}_{k+1}, \xi) \subset \mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$ be the subset of those K whose restriction to $B_{\mathbf{I}}$ belongs to the set $\mathcal{K}_{\mathbf{I}}^*$ for each $\mathbf{I}$ with $|\mathbf{I}| \geq 4$. Extending sections from $B_{\mathbf{I}}$ to $\overline{\mathcal{M}}_{k+1}^* \times M$ with support in $\overline{\mathcal{M}}_{k+1}^\delta \times M$ as in [20], we see that the restriction map to $B_{\mathbf{I}}$ is continuous and open. Since the preimage of a Baire subset under a continuous open map is a Baire subset, we conclude that $\mathcal{K}^*(\overline{\mathcal{M}}_{k+1}, \xi)$ is Baire in $\mathcal{K}(\overline{\mathcal{M}}_{k+1}, \xi)$.

We claim that the set $\mathcal{K}^*(\overline{\mathcal{M}}_{k+1}, \xi)$ has the desired property. Arguing by contradiction, suppose that there exists $K \in \mathcal{K}^*(\overline{\mathcal{M}}_{k+1}, \xi)$, $\mathbf{z} \in \overline{\mathcal{M}}_k$, $U \subset \Sigma_{\mathbf{z}}^\delta$ open nonempty, a $K_{\mathbf{z}}$-holomorphic map $f = (a, u) : U \to \mathbb{R} \times M$, and an open disk $D \subset U$ on which $\pi_\xi du \equiv 0$. This implies that the image $u(D)$ is contained in a single Reeb orbit $\gamma \subset M$. Moreover, from Eq. (14) it follows that $K_{\mathbf{z}}(z, u(z)) = 0$ for all $z \in D$. Hence the graph $\widetilde{D} := \{(\mathbf{z}, z, u(z)) \mid z \in D\}$ of $u|_D$ is an embedded 2-disk contained in the intersection of the zero set $K^{-1}(0)$ with $\Sigma_{\mathbf{z}}^\delta \times \gamma$.

Now the disk D is contained in some stratum $\overline{\mathcal{M}}_{\mathbf{I}}^*$ such that $\mathbf{z} \in \pi(\overline{\mathcal{M}}_{\mathbf{I}}^*)$, and $|\mathbf{I}| \geq 4$ because D is contained in the δ-thick part. By construction, the restriction $K_{\mathbf{I}}$ of K to the stratum $B_{\mathbf{I}}$ belongs to the Baire set $\mathcal{K}_{\mathbf{I}}^*$. But by the previous paragraph the 2-disk $\widetilde{D}$ is contained in the intersection of $K_{\mathbf{I}}^{-1}(0) \cap B_{\mathbf{I},0}$ with the leaf $\Sigma_{\mathbf{z}}^* \times \gamma$ of the foliation $\mathcal{F}_I$, contradicting the property of the Baire set in Proposition E.1. □

7 Linearization at punctured K-holomorphic maps

In this section we study the linearization of the universal Cauchy-Riemann operator at punctured K-holomorphic maps into symplectic cobordisms. Our goal is to derive conditions under which this linearization is surjective. Throughout this section we make the following assumptions.

- X is a symplectic cobordism with cylindrical ends $\mathbb{R}_+ \times \overline{M}, \mathbb{R}_- \times \underline{M}$ modelled over contact manifolds $\overline{M}$, $\underline{M}$, equipped with a compatible almost complex structure J. We also allow for X to be a symplectization $\mathbb{R} \times M$, which we formally include by letting $X_0 = \emptyset$ and $\overline{M} = \underline{M} = M$.
- The manifolds $\overline{M}$, $\underline{M}$ are of Morse–Bott type; we denote by $\overline{N}$ the Bott manifold of parametrized closed Reeb orbits on $\overline{M}$ (of all periods, possibly multiply covered) and by $\overline{N}/S^1$ its quotient by reparametrization, and similarly for $\underline{M}$.
- All the Morse–Bott families $\overline{N}/S^1$ and $\underline{N}/S^1$ are manifolds (not orbifolds).

Here the last assumption is not really necessary, but it simplifies the discussion and will be satisfied in the situations considered in this paper.

7.1 Fredholm setup

We adapt the Fredholm setup for moduli spaces of punctured holomorphic curves in [13] to our situation. We consider first the case of a fixed domain and fix the following data:

- distinct points $\overline{z} := (\overline{z}_1, \dots, \overline{z}_{\overline{p}})$, $\underline{z} := (\underline{z}_1, \dots, \underline{z}_{\underline{p}})$ on the Riemann sphere $S = S^2$, denoting by $\dot{S} := S \backslash \{\overline{z}_i, \underline{z}_j\}$ the corresponding punctured sphere; we assume that $\overline{p} + \underline{p} \geq 3$ and denote by $\dot{S}^\delta$ the δ-thick part of the hyperbolic surface $\dot{S}$;
- holomorphic cylindrical coordinates $\mathbb{R}_\pm \times S^1$ near the positive and negative punctures of $\dot{S}$, respectively;
- tuples $\overline{\Gamma} := (\overline{\Gamma}_1, \dots, \overline{\Gamma}_{\overline{p}})$, $\underline{\Gamma} := (\underline{\Gamma}_1, \dots, \underline{\Gamma}_{\underline{p}})$ where $\overline{\Gamma}_i = \overline{N}_i/S^1$, $\underline{\Gamma}_i = \underline{N}_i/S^1$ for connected components $\overline{N}_i \subset \overline{N}$, $\underline{N}_i \subset \underline{N}$ in the manifolds of closed Reeb orbits on $\overline{M}$, $\underline{M}$;
- $m \in \mathbb{N}$, $p > 2$, and a small $d > 0$.

Note that here we consider only spheres with at least 3 punctures; planes and cylinders will be dealt with later by separate arguments (for ordinary J-holomorphic curves because $K \equiv 0$ in these cases). We denote by

$$\mathcal{B} := \mathcal{B}^{m,p,d}(X, \overline{\Gamma}, \underline{\Gamma})$$

the space of all maps $f : \dot{S} \to X$ of Sobolev class $W^{m,p}_{\mathrm{loc}}$ which converge near each positive puncture $\overline{z}_i$ to a closed Reeb orbit $\overline{\gamma}_i$ in the family $\overline{\Gamma}_i$ in the following sense (and similarly near the negative punctures): Suppose first that $\overline{\gamma}_i$ is simple. Pick a tubular neighbourhood $V \cong \mathbb{R}/\overline{T}_i\mathbb{Z} \times \mathbb{R}^{\dim \overline{\Gamma}_i} \times \mathbb{R}^{2n-\dim \overline{\Gamma}_i - 2}$ of $\overline{\gamma}_i \cong \mathbb{R}/\overline{T}_i\mathbb{Z} \times \{\overline{v}_i\} \times \{0\}$ in $\overline{M}$ such that $\overline{\Gamma}_i \cap V \cong \mathbb{R}/\overline{T}_i\mathbb{Z} \times \mathbb{R}^{\dim \overline{\Gamma}_i} \times \{0\}$. Then f maps a neighbourhood $[\rho, \infty) \times S^1$ of $\overline{z}_i$ to $\mathbb{R} \times V$ such that in the cylindrical coordinates $(s, t) \in [\rho, \infty) \times S^1$ near $\overline{z}_i$ the map $f = (a, \vartheta, v, w) : [\rho, \infty) \times S^1 \to \mathbb{R} \times \mathbb{R}/\overline{T}_i\mathbb{Z} \times \mathbb{R}^{\dim \overline{\Gamma}_i} \times \mathbb{R}^{2n-\dim \overline{\Gamma}_i - 2}$ satisfies

$$\Big(a(s,t) - \overline{a}_i - \overline{T}_i s, \vartheta(s,t) - \overline{\vartheta}_i - \overline{T}_i t, v(s,t) - \overline{v}_i, w(s,t)\Big) \\ \in W^{m,p,d}\big([\rho,\infty) \times S^1, \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\dim \overline{\Gamma}_i} \times \mathbb{R}^{2n-\dim \overline{\Gamma}_i - 2}\big)$$

for constants $\overline{a}_i \in \mathbb{R}$, $\overline{\vartheta}_i \in \mathbb{R}/\overline{T}_i\mathbb{Z}$. Here $W^{m,p,d}\big([\rho,\infty) \times S^1, \mathbb{R}^{2n}\big)$ denotes the weighted Sobolev space of functions g such that $e^{ds}g(s,t)$ belongs to $W^{m,p}$. An open neighbourhood of f is given by maps $f' : \dot{S} \to X$ of Sobolev class $W^{m,p}_{\mathrm{loc}}$, which map a neighbourhood $[\rho, \infty) \times S^1$ of $\overline{z}_i$ to $\mathbb{R} \times V$ such that

$$\Big(a'(s,t) - \overline{a'}_i - \overline{T}_i s, \vartheta'(s,t) - \overline{\vartheta'}_i - \overline{T'}_i t, v'(s,t) - \overline{v'}_i, w'(s,t)\Big) \\ \in W^{m,p,d}\big([\rho,\infty) \times S^1, \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\dim \overline{\Gamma}_i} \times \mathbb{R}^{2n-\dim \overline{\Gamma}_i - 2}\big)$$

for constants $\overline{a'}_i \in \mathbb{R}$, $\overline{\vartheta'}_i \in S^1$, and $\overline{v'}_i \in \mathbb{R}^{\dim \overline{\Gamma}_i}$. The case of k-fold covered $\overline{\gamma}_i$ is reduced to the previous one by passing to the k-fold cover of the neighbourhood V, see [12]. Then $\mathcal{B}$ is a Banach manifold with a bundle projection

$$\mathrm{Ev} : \mathcal{B} \to \prod_{i=1}^{\overline{p}} \overline{\Gamma}_i \times \prod_{j=1}^{\underline{p}} \underline{\Gamma}_j, \qquad f \mapsto (\overline{v}_1, \dots, \overline{v}_{\overline{p}}, \underline{v}_1, \dots, \underline{v}_{\underline{p}}).$$

Note that the Banach manifold $\mathcal{B}$ and the bundle projection Ev do not depend on the chosen cylindrical coordinates near the punctures.

We will fix J (and often suppress it from the notation), and vary K in the space

$$\mathcal{K}_{\dot{S}} \subset \mathcal{K}(\dot{S}, TX)$$

of perturbations obtained by restricting coherent perturbations as in Definition 6.4 satisfying $\|K\| < 1$ to an irreducible component of a fibre of the fibration $\overline{\mathcal{M}}_{k+1} \to \overline{\mathcal{M}}_k$. (Hence $\dot{S}$ is a sphere with at least three punctures.)

Recall that K is required to be $\mathbb{R}$-invariant and take values in $\overline{\xi}$ or $\underline{\xi}$, respectively, outside a compact subset $X_0 \subset X$ (and on all of $X = \mathbb{R} \times M$ in the cylindrical case). Since K has support in $\dot{S}^\delta$ by definition, K-holomorphic maps $\dot{S} \to X$ are just J-holomorphic near the punctures. In view of the asymptotics in [41], each punctured K-holomorphic map $f : \dot{S} \to X$ asymptotic to $\overline{\Gamma}$, $\underline{\Gamma}$ belongs to $\mathcal{B}^{m,p,d}$ for d sufficiently small. Let

$$\mathcal{E} := \mathcal{E}^{m-1,p,d} \to \mathcal{B}$$

be the Banach bundle whose fibre at $f \in \mathcal{B}$ is given by

$$\mathcal{E}_f = W^{m-1,p,d}\big(\dot{S}, \mathrm{Hom}^{0,1}(T\dot{S}, f^*TX)\big).$$

Then for $K \in \mathcal{K}_{\dot{S}}$ the Cauchy-Riemann operator defines a smooth section

$$\bar{\partial}_{J,K} : \mathcal{B} \to \mathcal{E}, \qquad f \mapsto \bar{\partial}_{J,K} f.$$

We denote its linearization by

$$D_f : T_f\mathcal{B} \to \mathcal{E}_f.$$

More generally, the *universal Cauchy-Riemann operator* is the section

$$\bar{\partial} : \mathcal{B} \times \mathcal{K}_{\dot{S}} \to \mathcal{E}, \qquad (f, K) \mapsto \bar{\partial}_{J,K} f.$$

Its linearization

$$T_f\mathcal{B} \oplus \mathcal{K}_{\dot{S}} \to \mathcal{E}_f$$

can be described as follows. Recall that f maps a neighbourhood $[\overline{\rho}_i, \infty) \times S^1$ of the puncture $\overline{z}_i$ into $\mathbb{R} \times V$ with V as defined above. Let $\overline{\zeta}_i \in \mathbb{R}^{2+\dim \overline{\Gamma}_i}$ be a tangent vector to $\mathbb{R} \times \mathbb{R}/\overline{T}_i\mathbb{Z} \times \mathbb{R}^{\dim \overline{\Gamma}_i}$ at $(\overline{a}_i, \overline{\vartheta}_i, \overline{v}_i)$. For each i pick a cutoff function $\overline{\beta}_i : \mathbb{R} \to [0, 1]$ with $\overline{\beta}_i(s) = 1$ for $s \geq \overline{\rho}_i + 1$ and $\overline{\beta}_i(s) = 0$ for $s \leq \overline{\rho}_i$. Then $\overline{\zeta}_i$ extends to a section $\overline{\beta}_i(s)\overline{\zeta}_i$ in f^*TX and we get an isomorphism

$$\bigoplus_{i=1}^{\overline{p}} \mathbb{R}^{2+\dim \overline{\Gamma}_i} \oplus \bigoplus_{j=1}^{\underline{p}} \mathbb{R}^{2+\dim \underline{\Gamma}_j} \oplus W^{m,p,d}(f^*TX) \to T_f\mathcal{B}, \qquad (\zeta, g) \mapsto \hat{\zeta} + g, \tag{15}$$

where for $\zeta = (\overline{\zeta}_i, \underline{\zeta}_j)$ we have set

$$\hat{\zeta} := \sum_i \overline{\beta}_i(s)\overline{\zeta}_i + \sum_j \underline{\beta}_j(s)\underline{\zeta}_j.$$

Under this isomorphism, the linearization of the universal Cauchy-Riemann operator at (f, K) is given by

$$\begin{aligned}\bigoplus_{i=1}^{\overline{p}} \mathbb{R}^{2+\dim \overline{\Gamma}_i} \oplus \bigoplus_{j=1}^{\underline{p}} \mathbb{R}^{2+\dim \underline{\Gamma}_j} \oplus W^{m,p,d}(f^*TX) \oplus \mathcal{K}_{\dot{S}} \to \mathcal{E}_f, \\ (\zeta, g, Y) \mapsto D_f g + D_f \hat{\zeta} + Y(z, f).\end{aligned} \tag{16}$$

7.2 Surjectivity of the linearized operator

Following [30], for a Euclidean vector bundle $F \to B$ over a compact manifold B (possibly with boundary and corners) we denote the space of *Floer's* C^ε-*sections* in F by

$$C^\varepsilon(B, F) := \left\{ s : B \to F \text{ smooth section} \;\middle|\; \sum_{i=1}^{\infty} \varepsilon_i \|s\|_{C^i} < \infty \right\}.$$

Here $\varepsilon = (\varepsilon_i)_{i \in \mathbb{N}}$ is a fixed sequence of positive numbers and $\| \;\|_{C^i}$ is the C^i-norm with respect to some connection on F. It is shown in Lemma 5.1 of [30] that if the ε_i converge sufficiently fast to zero, then $C^\varepsilon(B, F)$ is a Banach space consisting of smooth sections and containing sections with support in arbitrarily small sets in B.

Remark 7.1 We cannot replace the space C^ε by C^∞ because the latter one is not a Banach space and we need to apply the implicit function theorem below. However, we could replace C^ε by C^N for sufficiently large N (which amounts to choosing $\varepsilon_i = 0$ for $i > N$); then the moduli spaces of holomorphic spheres would not be smooth but of class C^N as well, which would suffice for the applications in this paper.

The following lemma is an adaptation of the corresponding result in [20], using a result of Wendl [67] to deal with the cylindrical case. Denote by

$$\mathcal{K}^\varepsilon_{\dot{S}} \subset \mathcal{K}_{\dot{S}}$$

the subspace of Floer's C^ε-sections with support in $\dot{S}^\delta$, which is a Banach space equipped with the C^ε-norm. Recall that (X, J) is an almost complex manifold of dimension $2n$ with cylindrical ends modelled over contact manifolds satisfying the conditions at the beginning of this section.

Lemma 7.2 *In the notation above, let* $K \in \mathcal{K}^\varepsilon_{\dot{S}}$ *and* $f \in \mathcal{B} = \mathcal{B}^{m,p,d}(X, \overline{\Gamma}, \underline{\Gamma})$ *satisfy* $\bar{\partial}_{J,K} f = 0$ *as well as the following condition:*

(S) *If $f(\dot{S}^\delta)$ is entirely contained in a cylindrical end modelled over a contact manifold (M, ξ), then $\pi_\xi du(z) \neq 0$ for some z in the interior of $\dot{S}^\delta$, where $f = (a, u)$.*

Then the linearization of

$$\bar{\partial} \times \mathrm{Ev} : \mathcal{B} \times \mathcal{K}^\varepsilon_{\dot{S}} \to \mathcal{E} \times \prod_{i=1}^{\overline{p}} \overline{\Gamma}_i \times \prod_{j=1}^{\underline{p}} \underline{\Gamma}_j$$

at (f, K) is surjective.

Proof Note that under the isomorphism (15), the linearized evaluation map is simply the projection $(\zeta, g, Y) \mapsto P\zeta$, where

$$P : \bigoplus_{i=1}^{\overline{p}} \mathbb{R}^{2+\dim \overline{\Gamma}_i} \oplus \bigoplus_{j=1}^{\underline{p}} \mathbb{R}^{2+\dim \underline{\Gamma}_j} \to \bigoplus_{i=1}^{\overline{p}} \mathbb{R}^{\dim \overline{\Gamma}_i} \oplus \bigoplus_{j=1}^{\underline{p}} \mathbb{R}^{\dim \underline{\Gamma}_j}$$

is the projection onto the directions of the Bott manifolds $\overline{\Gamma}_i, \underline{\Gamma}_j$. Hence, in view of (16), the assertion of the lemma is equivalent to surjectivity of the map

$$L : (\mathbb{R}^2)^{\overline{p}+\underline{p}} \oplus W^{m,p,d}(f^*TX) \oplus \mathcal{K}^\varepsilon_{\dot{S}} \to \mathcal{E}_f,$$
$$(\zeta, g, Y) \mapsto D_f(\hat{\zeta} + g) + Y(z, f).$$

The proof of surjectivity of L follows the line of the proof of [20, Lemma 4.1]. We will prove surjectivity of L for $m = 1$; the general case then follows by elliptic regularity.

Since L is a Fredholm operator (here we use again that K has compact support), it suffices to show that the orthogonal complement of its image is trivial. Suppose that η is an $L^{q,-d}$-section orthogonal to the image of L (where $1/p + 1/q = 1$), so η satisfies the equations

$$\begin{aligned} \langle D_f(\hat{\zeta} + g), \eta\rangle_{L^2} &= 0, \\ \langle Y, \eta\rangle_{L^2} &= 0 \end{aligned} \tag{17}$$

for all $(\zeta, g, Y) \in (\mathbb{R}^2)^{\overline{p}+\underline{p}} \oplus W^{1,p,d}(f^*TX) \oplus \mathcal{K}_{\dot{S}}$. The first equation (with $\zeta = 0$) implies that η satisfies a Cauchy-Riemann type equation $D_f^*\eta = 0$, so η is smooth and satisfies unique continuation. Now we distinguish two cases.
Case 1: $f(\dot{S}^\delta)$ intersects the interior of X_0.
Then Y can take arbitrary values in TX over $U_0 := \dot{S}^\delta \cap f^{-1}(X_0)$ and the second equation in (17) implies $\eta|_{U_0} \equiv 0$. By unique continuation, it follows that η must vanish identically on $\dot{S}$.

Case 2: $f(\dot{S}^\delta)$ is entirely contained in a cylindrical end.
Then f maps $\dot{S}^\delta$ to a cylindrical manifold $\mathbb{R}\times M$ (this includes the case that the original manifold X was a symplectization), and the argument of Case 1 does not work directly because Y can only take values in the distribution $\xi \subset TM$. Instead, we use condition (S) in the lemma according to which there exists a nonempty open subset $U \subset \dot{S}^\delta$ such that $\pi_\xi du(z) \neq 0$ for all $z \in U$, where $f = (a, u)$ over U. We will show surjectivity of the operator L restricted to the subspace in which the factor $\mathcal{K}^\varepsilon_{\dot{S}}$ is replaced by the subspace $\mathcal{K}^\varepsilon_U \subset \mathcal{K}^\varepsilon_{\dot{S}}$ of sections with support in $\overline{U}$. Consider the splitting of the complex vector bundle $\big(T(\mathbb{R}\times M), J\big)$ into complex subbundles

$$T(\mathbb{R}\times M) \cong \underline{\mathbb{C}} \oplus \xi, \qquad \underline{\mathbb{C}} = \mathbb{R}\partial_r \oplus \mathbb{R}R.$$

We restrict Eq. (17) to smooth sections $g \in C_0^\infty(U, f^*TX)$ with support in U. On such sections, with respect to the splitting above the linearized Cauchy-Riemann operator at f has the form

$$D_f \begin{pmatrix} h \\ w \end{pmatrix} = \begin{pmatrix} Ah + Bw \\ Ch + Dw \end{pmatrix}$$

with linear Cauchy-Riemann operators A and D on the bundles $f^*\underline{\mathbb{C}}$ and $f^*\xi$, respectively, and zero order operators B, C. Here $h \in C_0^\infty(U, f^*\underline{\mathbb{C}})$ and $w \in C_0^\infty(U, f^*\xi)$. According to [67, Lemma 8.10], the zero order operator $B : C_0^\infty(U, f^*\xi) \to C_0^\infty\big(U, \mathrm{Hom}^{0,1}(TU, f^*\underline{\mathbb{C}})\big)$ at $f = (a, u)$ is given by

$$Bw = -d\lambda(w, J\pi_\xi du)\partial_r + d\lambda(w, \pi_\xi du)R. \tag{18}$$

[To see this, apply [67, Lemma 8.10] to the $\widehat{J}$-holomorphic graph $\widehat{f} = \mathrm{id}\times f : U \to U \times (\mathbb{R}\times M)$, with $\widehat{J}$ and the stable Hamiltonian structure $(\widehat{\omega}, \widehat{\lambda})$ as defined in Eqs. (13) and (12); since $\widehat{\lambda} = \lambda$, the corresponding operator $\widehat{B}$ satisfies $\widehat{B}(0, w) = Bw$ with B given by (18).]

Now we invoke the hypothesis that $\overline{M}$ and $\underline{M}$ are contact manifolds. Hence λ is a contact form, so $(d\lambda, J)$ defines a Hermitian structure on ξ. Recall that for each $z \in U$ we have $\pi_\xi du(z) \neq 0$, so there exists $v \in T_zU$ with $\pi_\xi du(z)\cdot v \neq 0$. After rescaling v, we can therefore find a unitary basis $e_1, \dots, e_{n-1}$ of $(\xi_z, J, d\lambda)$ with $e_1 = \pi_\xi du(z)\cdot v$. Then at the point z we have $Be_1 \cdot v = -\partial_r$ and $BJe_1 \cdot v = -R$, which shows that the pointwise operator B is surjective at each $z \in U$.

Writing $\eta = (\rho, \sigma)$ with respect to the splitting over U, Eq. (17) become

$$\begin{aligned} \langle A(\hat{\zeta} + h), \rho \rangle_{L^2} + \langle C(\hat{\zeta} + h), \sigma \rangle_{L^2} &= 0, \\ \langle Bw, \rho \rangle_{L^2} + \langle Dw, \sigma \rangle_{L^2} &= 0, \\ \langle Y, \sigma \rangle_{L^2} &= 0 \end{aligned} \tag{19}$$

for all $(\zeta, h, w, Y) \in \mathbb{C}^{\overline{p}+\underline{p}} \oplus C_0^\infty(U, f^*\underline{\mathbb{C}}) \oplus C_0^\infty(U, f^*\xi) \oplus \mathcal{K}_U$. Since Y can vary freely in the ξ directions, the third equation implies that $\sigma|_U \equiv 0$. In view of the pointwise surjectivity of B on U, the second equation then implies that $\rho|_U \equiv 0$. Thus $\eta|_U \equiv 0$, and unique continuation yields $\eta|_{\dot{S}} \equiv 0$. This concludes the proof of Lemma 7.2. □

7.3 Holomorphic cylinders

Finally, let us record a lemma for (unperturbed) holomorphic cylinders that we will need in the proof of Theorem 1.2 (b). Let T^*T^n be the cotangent bundle of the n-torus, equipped with an almost complex structure J_ρ induced by the standard flat metric on T^n,

$$J_\rho : \frac{\partial}{\partial q_i} \mapsto -\rho(|p|)\frac{\partial}{\partial p_i}, \qquad \rho(|p|)\frac{\partial}{\partial p_i} \mapsto \frac{\partial}{\partial q_i}.$$

Here (q_i, p_i), $i = 1, \ldots, n$, are standard coordinates on $T^*T^n \cong T^n \times \mathbb{R}^n$ and $\rho : [0, \infty) \to (0, \infty)$ is a smooth function satisfying $\rho(r) = 1$ near $r = 0$ and $\rho(r) = r$ for large r.

Lemma 7.3 *Let $\sigma \in H_1(T^n; \mathbb{Z})$ be the k-fold multiple of a primitive class in $H_1(T^n; \mathbb{Z})$. Then the moduli space $\mathcal{M}_\sigma$ of J_ρ-holomorphic cylinders in T^*T^n with two positive punctures asymptotic to closed geodesics in the classes $\pm\sigma$ is a smooth manifold of dimension $2n - 2$ whose image under the evaluation map is a k-fold covering of T^*T^n.*

Proof Write $\sigma = k\bar{\sigma}$ for a primitive class $\bar{\sigma} \in H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$ and $k \geq 1$. Note that a linear coordinate change $(q, p) \mapsto (Aq, Ap)$ does not change the form of J_ρ (only the norm $|p|$ looks different in the new coordinates). So after such a coordinate change we may assume that $\bar{\sigma} = (1, 0, \ldots, 0)$.

Note that for each constant $(\bar{q}', \bar{p}') = (\bar{q}_2, \ldots, \bar{q}_n, \bar{p}_2, \ldots, \bar{p}_n)$ the *orbit cylinder*

$$C_{(\bar{q}', \bar{p}')} := \{(q, p) \in T^*T^n \mid q_i = \bar{q}_i, p_i = \bar{p}_i, i = 2, \ldots, n\} \subset T^*T^n$$

is J_ρ-invariant, and these orbit cylinders foliate T^*T^n. Each orbit cylinder is the image of a J_ρ-holomorphic k-fold covering map $f : \mathbb{R} \times S^1 \to C_{(\bar{q}', \bar{p}')}$, which is unique up to automorphism of the domain. To see this, let us write out the equations for J_ρ-holomorphicity of a map $f = (q, p) : \mathbb{R} \times S^1 \to T^*T^n$:

$$\partial_t q_i = \frac{1}{\rho}\partial_s p_i, \qquad \partial_s q_i = -\frac{1}{\rho}\partial_t p_i, \qquad i = 1, \dots, n. \tag{20}$$

So we obtain a J_ρ-holomorphic k-fold covering map $f = (q, p) : \mathbb{R} \times S^1 \to C_{(\bar{q}', \bar{p}')}$ by setting $q_i = \bar{q}_i$, $p_i = \bar{p}_i$ for $i = 2, \dots, n$, $q_1(s, t) := kt + \bar{q}_1$, and defining $p_1(s, t) = p_1(s)$ to be the solution of $\partial_s p_1 = \rho\big(|(p_1(s), \bar{p}')|\big)$ with $p_1(0) = \bar{p}_1$. The conditions on ρ ensure that p_1 defines a diffeomorphism $\mathbb{R} \to \mathbb{R}$. Hence f is a J_ρ-holomorphic cylinder with two positive punctures asymptotic to the closed geodesics $t \mapsto (\pm kt, \bar{q}')$ in the classes $\pm\sigma$. Varying the constants $(\bar{q}', \bar{p}')$, these cylinders form a smooth manifold of dimension $2n - 2$ whose image under the evaluation map is a k-fold covering of T^*T^n. So we are done once we show that the moduli space $\mathcal{M}_\sigma$ consists only of the k-fold orbit cylinders.

Consider an arbitrary J_ρ-holomorphic cylinder $f = (q, p) : \mathbb{R} \times S^1 \to T^*T^n$ belonging to $\mathcal{M}_\sigma$. It is asymptotic at its punctures to closed geodesics $t \mapsto (\pm kt, q^\pm)$ for constants $q^\pm = (q_2^\pm, \dots, q_n^\pm)$. Suppose first that $q^+ \neq q^-$, say $q_n^+ \neq q_n^-$. Pick a point $s^* \in \mathbb{R}$ at which $f(s^*, 0) = (q^*, p^*)$ with $q_n^* \neq q_n^\pm$. The codimension 2 hypersurface

$$Y := \{(q, p) \in T^*T^n \mid q_n = q_n^*, p_n = p_n^*\} = \bigcup_{\bar{q}_n = q_n^*, \bar{p}_n = p_n^*} C_{(\bar{q}', \bar{p}')}$$

is a union of orbit cylinders and J_ρ-invariant. Since Y lies over the set $\{q_n = q_n^*\}$ and f lies over the set $\{q_n = q_n^\pm\}$ at infinity, the image of f is disjoint from Y at infinity and the signed count of intersection points gives a well-defined homological intersection number $[f] \cdot [Y] \in \mathbb{Z}$. Since f and Y intersect by construction and are both J_ρ-holomorphic, positivity of intersection implies $[f] \cdot [Y] > 0$. On the other hand, we can deform $Y = Y_0$ through the hypersurfaces $Y_r := \{q_n = q_n^*, p_n = p_n^* + r\}$ to Y_R for large R. The preceding argument shows that intersections with f remain in a compact region during this deformation, hence $[f] \cdot [Y_R] = [f] \cdot [Y] > 0$. Now on each compact subset $K \subset \mathbb{R} \times S^1$, the component p_n of f is uniformly bounded and thus not equal to $p_n^* + R$ for large R. Hence for large R, intersections of f with Y_R can only occur near infinity in $\mathbb{R} \times S^1$, where they are excluded by the difference of the components q_n. So we conclude $[f] \cdot [Y_R] = 0$ for large R, contradicting our previous assertion. This proves that the constants $q^\pm$ must coincide, i.e., f is asymptotic at its punctures to closed geodesics $t \mapsto (\pm kt, \bar{q}')$ for a constant $\bar{q}' = (\bar{q}_2, \dots, \bar{q}_n)$.

The asymptotic behaviour shows that for $i = 2, \dots, n$ the component q_i of f defines a function $q_i : \mathbb{R} \times S^1 \to \mathbb{R}$. Adding up partial derivatives of Eqs. (20), we see that q_i satisfies the equation

$$\Delta q_i + \frac{1}{\rho}(\partial_s \rho\, \partial_s q_i + \partial_t \rho\, \partial_t q_i) = 0.$$

Since $q_i(s,t) \to \bar{q}_i$ as $s \to \pm\infty$, the maximum principle implies that q_i is constant equal to $\bar{q}_i$. Equation (20) shows that p_i is constant as well, hence f defines a k-fold covered parametrization of an orbit cylinder $C_{(\bar{q}', \bar{p}')}$. □

7.4 Broken punctured K-holomorphic maps

Next we generalize the preceding discussion to broken punctured K-holomorphic maps. We fix the following data:

- a split manifold $X^* = X^+ \amalg (\mathbb{R} \times M) \amalg X^-$ with compatible almost complex structure $J^* = (J^+, J_M, J^-)$, where M is a contact manifold of Morse–Bott type on which all Morse–Bott families N/S^1 are manifolds;
- a stable nodal curve $\mathbf{z}$ of genus zero modelled over the k-labelled tree T;
- for each vertex $\alpha \in T$ a component X_α of X^*;
- asymptotic data $\Gamma^* = (\{\Gamma_{\alpha\beta}\}_{\alpha E \beta}, \{\Gamma_i\}_{1 \le i \le k})$, where each $\Gamma_{\alpha\beta}$ (or Γ_i, respectively) is either
  - (i) a Morse–Bott component of closed Reeb orbits in M, or
  - (ii) the manifold X_α (or X_{α_i}, respectively).

We denote by $\mathcal{B}_T(X^*, \Gamma^*) = \prod_{\alpha \in T} \mathcal{B}_\alpha$ the space of collections $\mathbf{f} = (f_\alpha)_{\alpha \in T}$ of maps $f_\alpha : \dot{S}_\alpha \to X_\alpha$ of Sobolev class $W^{m,p,d}$ which near each puncture are asymptotic to the corresponding Morse–Bott component of closed Reeb orbits in case (i), or extend continuously over the puncture in case (ii).

Recall the space $\mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX)$ of coherent perturbations K on the split manifold X^* from Definiton 6.4. We denote by

$$\mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX) \subset \mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX)$$

the subspace consisting of Floer's C^ε-sections, which is a separable Banach space equipped with the C^ε-norm. We get a universal Cauchy-Riemann operator

$$\bar{\partial} : \mathcal{B}_T(X^*, \Gamma^*) \times \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX^*) \to \mathcal{E}, \qquad (\mathbf{f}, K) \mapsto \bar{\partial}_{J,K_{\mathbf{z}}} \mathbf{f} = (\bar{\partial}_{J,K_{\mathbf{z}}} f_\alpha)_{\alpha \in T}$$

with values in a suitable bundle $\mathcal{E}_T = \prod_{\alpha\in T}\mathcal{E}_\alpha \to \mathcal{B}_T = \prod_{\alpha\in T}\mathcal{B}_\alpha$. Moreover, we have an obvious evaluation map

$$\mathrm{Ev} : \mathcal{B}_T(X^*, \Gamma^*) \to \prod_{\alpha E\beta}\Gamma_{\alpha\beta} \times \prod_{1\le i\le k}\Gamma_i =: \prod_{\alpha\in T}\Gamma_\alpha,$$

where Γ_α denotes the product of the factors associated to the special points on α. For each $\alpha \in T$, we denote by $\mathcal{B}^s_\alpha \subset \mathcal{B}_\alpha$ the open subset consisting of those $f_\alpha : \dot{S}_\alpha \to X_\alpha$ satisfying condition (S) in Lemma 7.2. We define the open subset

$$\mathcal{B}^s_T(X^*, \Gamma^*) := \prod_{\alpha\in T}\mathcal{B}^s_\alpha \subset \mathcal{B}_T(X^*, \Gamma^*).$$

Proposition 7.4 *Let* (X^*, Γ^*) *and* $\mathbf{z}$ *be as above. Then the linearization of*

$$\bar\partial \times \mathrm{Ev} : \mathcal{B}^s_T(X^*, \Gamma^*) \times \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX^*) \to \mathcal{E}_T \times \prod_{\alpha E\beta}\Gamma_{\alpha\beta} \times \prod_{1\le i\le k}\Gamma_i$$

at each $(\mathbf{f}, K)$ *with* $\bar\partial_{J,K_\mathbf{z}}\mathbf{f} = 0$ *is surjective.*

Proof Removing the special points from the nodal surface $\Sigma_\mathbf{z}$ yields the disconnected punctured surface $\Sigma^*_\mathbf{z} = \amalg_{\alpha\in T}\dot{S}_\alpha$. So the space of coherent perturbations over $\Sigma^*_\mathbf{z}$ is the direct product

$$\mathcal{K}^\varepsilon(\Sigma^*_\mathbf{z}, TX^*) = \prod_{\alpha\in T}\mathcal{K}^\varepsilon(\dot{S}_\alpha, TX^*).$$

Combined with the preceding discussion, this shows that the linearization at $(\mathbf{f}, K)$ of the map

$$\bar\partial \times \mathrm{Ev} : \mathcal{B}^s_T(X^*, \Gamma^*) \times \mathcal{K}^\varepsilon(\Sigma^*_\mathbf{z}, TX^*) \to \mathcal{E}_T \times \prod_{\alpha E\beta}\Gamma_{\alpha\beta} \times \prod_{1\le i\le k}\Gamma_i$$

splits as a product of factors corresponding to $\alpha \in T$ and is therefore surjective by Lemma 7.2. In view of Lemma 6.3 (which continues to hold for Floer's C^ε-sections), this proves the proposition. □

Remark 7.5 Proposition 7.4 continues to hold if we replace the evaluation map at some removable puncture by an ℓ-jet evaluation map (picking the Sobolev class so high that it embeds into C^ℓ). This follows by combining the preceding proof with the proof of [20, Proposition 6.9].

7.5 Moduli spaces

Proposition 7.4 leads to regularity of moduli spaces by a standard argument. We keep fixed the tree T and the data X_α, Γ^*, but let $\mathbf{z}$ vary in the stratum $\mathcal{M}_T$ of Deligne–Mumford space. Let $A_\alpha \in H_2(X_\alpha, \Gamma_\alpha)$ be relative homology classes associated to the vertices $\alpha \in T$ and let $A := \sum_{\alpha\in T} A_\alpha$. For $K \in \mathcal{K}(\overline{\mathcal{M}}_{k+1}, TX^*)$ let $\mathcal{M}^{A_\alpha,K}(X_\alpha, \Gamma_\alpha)$ be the moduli space of punctured K-holomorphic spheres in X_α with asymptotics Γ_α and homology class A_α. Note that for $K = 0$ and in the absence of nodes this agrees with the moduli space $\mathcal{M}^{A_\alpha,J}(\overline{\Gamma}_\alpha, \underline{\Gamma}_\alpha)$ defined in Eq. (3) of Sect. 2 for the symplectic cobordism X_α and asymptotics $\Gamma_\alpha = \overline{\Gamma}_\alpha \cup \underline{\Gamma}_\alpha$. We denote by

$$\mathcal{M}_T^{A,K}(X^*, \Gamma^*) \subset \prod_{\alpha\in T} \mathcal{M}^{A_\alpha,K}(X_\alpha, \Gamma_\alpha)$$

the moduli space of broken punctured K-holomorphic spheres in X^* with asymptotics Γ^* and homology class A modelled over the tree T (whose asymptotics are required to match across edges of T).

We denote by $\Delta^E \subset \prod_{\alpha E\beta} \Gamma_{\alpha\beta}$ the product of the diagonals in $\Gamma_{\alpha\beta} \times \Gamma_{\beta\alpha}$ for $\alpha E\beta$. Moreover, we fix a submanifold $\mathcal{Z} \subset \prod_{1\le i\le k} \Gamma_i$. By Proposition 7.4, the map

$$\bar\partial \times \mathrm{Ev} : \mathcal{M}_T \times \mathcal{B}_T^s(X^*, \Gamma^*) \times \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX^*) \to \mathcal{E} \times \prod_{\alpha E\beta} \Gamma_{\alpha\beta} \times \prod_{1\le i\le k} \Gamma_i$$

is transverse to the submanifold $0_\mathcal{E} \times \Delta^E \times \mathcal{Z}$, where $0_\mathcal{E}$ denotes the zero section of $\mathcal{E}$. The implicit function theorem implies that the universal moduli space

$$\mathcal{U}_T^s := (\bar\partial \times \mathrm{Ev})^{-1}(0_\mathcal{E} \times \Delta^E \times \mathcal{Z})$$

is a Banach manifold. Consider the projection

$$\pi_\mathcal{U} : \mathcal{U}_T^s \to \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX^*)$$

onto the third factor. Let $\mathcal{K}^{\varepsilon,\mathrm{reg}}(\overline{\mathcal{M}}_{k+1}, TX^*) \subset \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX^*)$ be the set of regular values of this projection, as well as for the space with $\mathcal{Z}$ replaced by $\prod_{1\le i\le k} \Gamma_i$, as well as for all the corresponding spaces associated to subtrees of T. This set is Baire by the Sard-Smale theorem. Let $\mathcal{K}^{\varepsilon,*}(\overline{\mathcal{M}}_{k+1}, TX^*) \subset \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{k+1}, TX^*)$ be the subset of those K which restrict over the cylindrical piece $\mathbb{R}\times M$ to elements in the Baire subset $\mathcal{K}^{\varepsilon,*}(\overline{\mathcal{M}}_{k+1}, \xi)$ in Lemma 6.6 (which also holds within the space of Floer's C^ε-sections due to the last assertion in Proposition E.1). This set is a Baire subset because the restriction map is continuous and open. Therefore, the intersection

$$\mathcal{K}^{\varepsilon,\mathrm{reg},*}(\overline{\mathcal{M}}_{k+1}, TX^*) := \mathcal{K}^{\varepsilon,\mathrm{reg}}(\overline{\mathcal{M}}_{k+1}, TX^*) \cap \mathcal{K}^{\varepsilon,*}(\overline{\mathcal{M}}_{k+1}, TX^*)$$

is a Baire subset, in particular nonempty. Take some $K \in \mathcal{K}^{\varepsilon,\mathrm{reg},*}(\overline{\mathcal{M}}_{k+1}, TX^*)$. Now comes a crucial observation due to C. Wendl: according to Lemma 6.6, each component f_α of an element $(\mathbf{z}, \mathbf{f}) \in \mathcal{M}_T^{A,K}(X^*, \Gamma^*)$ satisfies condition (S) in Lemma 7.2, and therefore $(\mathbf{z}, \mathbf{f}, K) \in \mathcal{U}_T^s$. This shows that all components of punctured K-holomorphic spheres in the moduli space $\mathcal{M}_T^{A,K}(X^*, \Gamma^*)$ appear in manifolds of the expected dimensions, and the evaluation maps at the edges are transverse to the diagonal. Hence the total moduli space (taking preimages of the diagonals under the edge evaluation maps) is a smooth manifold of the expected dimension, and the evaluation map at the marked points is transverse to $\mathcal{Z}$. To work out the dimensions, we write

$$k = \overline{p} + \underline{p} + m, \qquad n_\alpha = \overline{p}_\alpha + \underline{p}_\alpha + m_\alpha,$$

where $\overline{p}, \underline{p}, m$ is the number of marked points corresponding to positive/negative punctures and marked points, respectively, and similarly for the number n_α of special points on the component α. We also write

$$|T| - 1 = e(T) = p(T) + m(T), \tag{21}$$

where $p(T), m(T)$ is the number of edges corresponding to punctures and nodes, respectively. We have the obvious relations

$$\sum_{\alpha\in T}(\overline{p}_\alpha + \underline{p}_\alpha) = \overline{p} + \underline{p} + 2p(T), \qquad \sum_{\alpha\in T} m_\alpha = m + 2m(T), \qquad \sum_{\alpha\in T} A_\alpha = A,$$

where A_α denotes the homology class of the map f_α. According to Eq. (4) now the dimension of the moduli space of punctured holomorphic spheres associated to the component α is given by

$$\begin{aligned}\dim \mathcal{M}^{A_\alpha,K}(X_\alpha, \Gamma_\alpha) = (n-3)(2 - \overline{p}_\alpha - \underline{p}_\alpha) + 2c_1(A_\alpha) + 2m_\alpha \\ + \sum_{i=1}^{\overline{p}_\alpha}\big(\mathrm{CZ}(\overline{\Gamma}_i^\alpha) + \dim \overline{\Gamma}_i^\alpha\big) - \sum_{j=1}^{\underline{p}_\alpha} \mathrm{CZ}(\underline{\Gamma}_j^\alpha).\end{aligned}$$

Taking preimages of the diagonals at the edges yields the dimension formula

$$
\begin{aligned}
\dim \mathcal{M}_T^{A,K}(X^*, \Gamma^*) &= \sum_{\alpha\in T} \dim \mathcal{M}^{A_\alpha,K}(X_\alpha, \Gamma_\alpha) - \frac{1}{2}\sum_{\alpha E\beta} \dim \Gamma_{\alpha\beta} \\
&= (n-3)(2|T|-2p(T)-\overline{p}-\underline{p})+2c_1(A)+2m+4m(T) \\
&\quad + \sum_{i=1}^{\overline{p}} \big(\mathrm{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i\big) - \sum_{j=1}^{\underline{p}} \mathrm{CZ}(\underline{\Gamma}_j) - 2nm(T) \\
&= (n-3)(2-\overline{p}-\underline{p}) + 2c_1(A) + 2m \\
&\quad + \sum_{i=1}^{\overline{p}} \big(\mathrm{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i\big) - \sum_{j=1}^{\underline{p}} \mathrm{CZ}(\underline{\Gamma}_j) - 2m(T).
\end{aligned}
$$

Here in the first line the factor $1/2$ appears because each edge contributes two terms to the sum over $\alpha E\beta$. For the second equality we use the dimension formula for $\mathcal{M}^{A_\alpha,K}(X_\alpha, \Gamma_\alpha)$ and the relations above. Here the terms $\mathrm{CZ}(\overline{\Gamma}_i^\alpha)$ and $\mathrm{CZ}(\underline{\Gamma}_j^\beta)$ corresponding to an edge $\alpha E\beta$ cancel, so only the Conley–Zehnder indices at punctures corresponding to marked points remain. The dimensions of the Bott manifolds in the second sum $-\frac{1}{2}\sum_{\alpha E\beta} \dim \Gamma_{\alpha\beta}$ corresponding to edges associated to punctures cancel with the corresponding terms in the first sum, while the terms $\dim \Gamma_{\alpha\beta} = 2n$ for edges $\alpha E\beta$ associated to nodal points appear only in the second sum, so we have

$$
\sum_{\alpha\in T} \sum_{i=1}^{\overline{p}_\alpha} \dim \overline{\Gamma}_i^\alpha - \frac{1}{2}\sum_{\alpha E\beta} \dim \Gamma_{\alpha\beta} = \sum_{i=1}^{\overline{p}} \dim \overline{\Gamma}_i - 2nm(T).
$$

The third equality follows from the computation

$$
\begin{aligned}
&(n-3)(2|T|-2p(T)) + 4m(T) - 2nm(T) \\
&= (n-3)(2+2m(T)) + 4m(T) - 2nm(T) = 2(n-3) - 2m(T).
\end{aligned}
$$

using Eq. (21). In summary, we have shown

Corollary 7.6 *For generic $K \in \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*)$, all components of punctured K-holomorphic spheres in the moduli space $\mathcal{M}_T^{A,K}(X^*, \Gamma^*)$ are manifolds of the expected dimensions, and the evaluation maps at the edges are transverse to the diagonal. Hence the total moduli space is a manifold of dimension*

$$\dim \mathcal{M}_T^{A,K}(X^*, \Gamma^*) = (n-3)(2-\overline{p}-\underline{p}) + 2c_1(A) + 2m \\ + \sum_{i=1}^{\overline{p}} \big(\mathrm{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i\big) - \sum_{j=1}^{\underline{p}} \mathrm{CZ}(\underline{\Gamma}_j) - 2m(T).$$

Moreover, we can arrange for the evaluation map on this moduli space at the marked points to be transverse to any given submanifold $\mathcal{Z} \subset \prod_{1\le i\le k} \Gamma_i$. □

8 Symplectic hypersurfaces in the complement of Lagrangian submanifolds

In order to apply the results of the previous section to our situation we need to put additional marked points on the domain such that all components of the limit of a Gromov–Hofer convergent sequence of holomorphic curves have stable domains. This will be achieved by adapting the ideas of [20] and Donaldson's construction of symplectic hypersurfaces to the situation at hand.

Consider a closed Lagrangian submanifold L in a closed symplectic manifold (X^{2n}, ω). Suppose that ω represents an integral relative cohomology class $[\omega] \in H^2(X, L; \mathbb{Z})$. Denote by $\mathrm{Tor}\, H_1(L; \mathbb{Z})$ the torsion subgroup of $H_1(L; \mathbb{Z})$ and by $|\mathrm{Tor}\, H_1(L; \mathbb{Z})| \in \mathbb{N}$ its order. Fix a compatible almost complex structure J on X. The following theorem is an easy adaptation of the main result in [6].

Theorem 8.1 (Auroux–Gayet–Mohsen [6]) *There exists a constant* $C > 0$ *such that for each sufficiently large integer multiple* D *of* $|\mathrm{Tor}\, H_1(L; \mathbb{Z})|$ *there exists a submanifold* $Y \subset X$ *of real codimension two, Poincaré dual to* $D[\omega]$*, such that its tangent space at each point differs from a complex subspace by an angle at most* $C D^{-1/2}$.

Remark 8.2 It follows directly from the definitions that for large D the submanifold Y constructed in Theorem 8.1 is symplectic, and $\bar{J}$-holomorphic for a compatible almost complex structure $\bar{J}$ arbitrarily C^0-close to J.

The proof of Theorem 8.1 is based on the following observation. Consider a compact oriented surface Σ with (possibly empty) boundary $\partial\Sigma$. Let $E \to \Sigma$ be a Hermitian line bundle together with a unitary trivialization $\tau : \partial\Sigma \times \mathbb{C} \to E|_{\partial\Sigma}$ over the boundary. We identify τ with the induced section $z \mapsto \tau(z, 1)$ of the unit circle bundle $P \subset E$ over $\partial\Sigma$. Define the *relative degree* of E with respect to the trivialization τ as

$$\deg(E, \tau) := \sum_{p \in s^{-1}(0)} \mathrm{ind}_p(s)$$

for any section s in E which is transverse to the zero section and coincides with τ along $\partial\Sigma$. The following relative version of the Chern–Weil theorem is proved exactly like the relative Gauss–Bonnet theorem (see e.g. [25]), which is the special case $E = T\Sigma$.

Lemma 8.3 *Let $A \in \Omega^1(P, i\mathbb{R})$ be a unitary connection 1-form on E with curvature $F_A \in \Omega^2(\Sigma, i\mathbb{R})$. Then*

$$\deg(E, \tau) = \frac{i}{2\pi}\left(\int_\Sigma F_A + \int_{\partial\Sigma} \tau^* A\right).$$

Note that the quantity $\int_{\partial\Sigma} \tau^* A$ measures the total deviation of the section τ from being parallel. In particular, if τ is parallel we have

$$\deg(E, \tau) = \frac{i}{2\pi}\int_\Sigma F_A.$$

Proof of Theorem 8.1 We follow the proof in [6] (for compatibility with the notation in [6], we will replace D by k). Fix a Hermitian line bundle $E \to X$ with first Chern class $[\omega]$ and and a Hermitian connection A on E with curvature form $F_A = -2\pi i\omega$. Let k be an integer multiple of $|\mathrm{Tor}\, H_1(L; \mathbb{Z})|$. The induced connection A_k on $E^{\otimes k}$ has curvature $F_{A_k} = -2\pi i k\omega$.

We first refine Lemma 2 in [6]. Let τ_k be a unitary section of $E^{\otimes k}|_L$. For any map $f : (\Sigma, \partial\Sigma) \to (X, L)$ from a compact surface Σ, Lemma 8.3 yields

$$\deg(f^* E^{\otimes k}, f^*\tau_k) = k\int_\Sigma \omega + \frac{i}{2\pi}\int_{\partial\Sigma} f^*\tau_k^* A_k.$$

In particular, the holonomy $\frac{i}{2\pi}\int_{\partial\Sigma} f^*\tau_k^* A_k$ is integral, so we can make it zero by a suitable change of τ_k. Applying this to all such maps f, we find a unitary section τ_k such that the closed 1-form $\tau_k^* A_k$ vanishes on the image of the map $\partial : H_2(X, L; \mathbb{Z}) \to H_1(L; \mathbb{Z})$. Arguing as in [6], we arrange for τ_k to satisfy the estimates in Lemma 2.

With this modification, the proofs of Lemmas 3 and 4, and thus of Theorem 2, in [6] work as before. Thus we find asymptotically holomorphic sections s_k of $E^{\otimes k}$ that are uniformly transverse to zero and do not vanish on L. Moreover, as remarked in Section 3.3 of [6], the sections satisfy $|\arg(s_k/\tau_k)| \le \pi/3$ on L. Hence we can deform s_k near L to arrange $s_k = \tau_k$ on L, without affecting its zero set $Y_k = s^{-1}(0)$. But then for any map $f : (\Sigma, \partial\Sigma) \to (X, L)$ from a compact surface we have

$$\deg(f^* E^{\otimes k}, f^*\tau_k) = \sum_{p \in s_k^{-1}(0)} \mathrm{ind}_p(s_k) = [Y_k] \cdot [f]$$

by definition of the degree. On the other hand, by the choice of τ_k above and Lemma 8.3, this degree equals $k \int_\Sigma \omega$, and Theorem 8.1 is proved. $\square$

9 Proof of transversality for Theorem 1.2(b)

In this section we prove the transversality needed for the proof of Theorem 1.2(b). We will begin in the more general setting of a Lagrangian $L \subset X$ and then specialize to the case $L = T^n$ and $X = \mathbb{C}P^n$.

The proof follows the scheme in [20]. Consider a closed symplectic manifold (X^{2n}, ω) and a closed Lagrangian submanifold $L \subset X$. After a small deformation of ω, we may assume that it represents a rational relative cohomology class $[\omega] \in H^2(X, L; \mathbb{Q})$, so $D_0[\omega] \in H^2(X, L; \mathbb{Z})$ for some positive integer D_0. We fix an energy $E > 0$, larger than the symplectic area of the holomorphic curves we want to consider (π in the case of Theorem 1.2(b)).

We fix some ω-compatible almost complex structure J_0 on X. Since the taming condition is open with respect to the C^0-norm $\| \;\|$ on X, there exists a constant $\theta_1 > 0$ with the following property: for every symplectic form $\bar{\omega}$ and every almost complex structure J satisfying $\|\bar{\omega} - \omega\| < \theta_1$ and $\|J - J_0\| < \theta_1$, the symplectic form $\bar{\omega}$ tames J.

Let us pick a tubular neighbourhood U of L. Since $H^2(X, L; \mathbb{R}) \cong H^2(X, U; \mathbb{R})$, we can represent a basis of $H^2(X, L; \mathbb{R})$ by closed forms with compact support in $X \backslash U$. It follows that there exists an open neighbourhood Ω of $[\omega]$ in $H^2(X, L; \mathbb{R})$ represented by symplectic forms $\bar{\omega}$ which agree with ω on U and satisfy $\|\bar{\omega} - \omega\| < \theta_1$. Thus for every almost complex structure J with $\|J - J_0\| < \theta_1$ and every J-holomorphic map $f : (\Sigma, \partial\Sigma) \to (X, L)$ from a compact Riemann surface with boundary,

$$\langle [\bar{\omega}], [f] \rangle = \int_\Sigma f^* \bar{\omega} \geq 0 \text{ for all } [\bar{\omega}] \in \Omega. \tag{22}$$

This condition for sufficiently many such $\bar{\omega}$ together with an energy bound $\langle [\omega], [f] \rangle < E$ defines the bounded intersection of a cone with a half-space. This shows that there is only a finite set $\mathcal{A}_E \subset H_2(X, L; \mathbb{Z})$ of homology classes that can be represented by such J-holomorphic maps f with $\|J - J_0\| < \theta_1$ and $\langle [\omega], [f] \rangle < E$.

Note that (22) continues to hold for every J^+-holomorphic map $f : \Sigma \to X \backslash L$ from a closed Riemann surface (of genus g) with p punctures asymptotic to closed geodesics $\gamma_1, \ldots, \gamma_p$ on L, where $\|J - J_0\| < \theta_1$ and J^+ is obtained by deforming J inside U to a cylindrical almost complex structure tamed by ω. Such holomorphic curves appear in moduli spaces of expected dimension

$$
\begin{aligned}
(n-3)(2-2g-p) - \sum_{i=1}^{p} \mathrm{CZ}(\gamma_i) &= (n-3)(2-2g-p) + \mu([f]) - \sum_{i=1}^{p} \mathrm{ind}(\gamma_i) \\
&\le (n-3)(2-2g-p) + \mu([f]).
\end{aligned}
$$

Here $\mathrm{CZ}(\gamma_i) = \mathrm{CZ}(\gamma_i, \Phi)$ is computed with respect to a trivialization of $\gamma_i^*\xi$ induced by a trivialization Φ of f^*TX over Σ. Then $\mu([f]) = \sum_{i=1}^{p} \mu(\gamma_i, \Phi)$ is the Maslov index of f and the equality follows from Lemma 2.1.

Since $[f]$ belongs to the finite set $\mathcal{A}_E$ from above, restricting to $g = 0$ and $p = 1$ we have shown

Lemma 9.1 *There exists a constant $D_E > 0$ with the following property: for every ω-tamed cylindrical almost complex structure J^+ satisfying $\|J^+ - J_0\| < \theta_1$ outside U, every nonempty moduli space of J^+-holomorphic planes in $X \backslash L$ asymptotic to L of energy $< E$ has expected dimension $\le D_E$.* □

By Theorem 8.1, for each sufficiently large integer multiple D of $D_0|\mathrm{Tor}\, H_1(L;\mathbb{Z})|$, there exists an approximately J_0-holomorphic closed codimension two submanifold $Y \subset X \backslash L$ whose Poincaré dual in $H^2(X, L; \mathbb{Z})$ equals $D[\omega]$. By choosing D large, we can make the maximal angle between a tangent space to Y and a J_0-complex subspace (the "Kähler angle") smaller than any given constant $\theta_2 > 0$. As shown in [20], for θ_2 sufficiently small there exists an almost complex structure J with $\|J - J_0\| < \theta_1$ such that Y is J-complex.

We fix a Riemannian metric on L such that the unit disk cotangent bundle D^*L embeds symplectically into $(X \backslash Y) \cap U$. We deform the ω-compatible almost complex structure J inside U to make it cylindrical near $M = \partial D^*L$, so it induces asymptotically cylindrical almost complex structures J^+ on $X^+ = X \backslash L$, J^- on $X^- = T^*L$, and J_M on $\mathbb{R} \times M$.

We fix a point $x \in L$ and the germ of a J-complex hypersurface Z at x. We perturb J such that all moduli spaces of *simple* punctured holomorphic curves for $J^\pm$ and J_M (including tangency conditions to Y and Z) as well as $J|_Y$ are transversely cut out. By [20, Proposition 6.9], tangency of order ℓ to Y (at some point which is not fixed) is a condition of codimension 2ℓ for simple curves. In view of Lemma 9.1, the maximal order of tangency to Y of simple J^+-holomorphic planes in $X \backslash L$ of energy $< E$ is therefore bounded by $D_E/2$. Since the homology classes of J^+-holomorphic planes in $X \backslash L$ belong to the finite set $\mathcal{A}_E \subset H_2(X, L; \mathbb{Z})$ from above, their multiplicity is bounded by some constant M_E and we have shown

Lemma 9.2 *The maximal order of tangency to Y of (not necessarily simple) J^+-holomorphic planes in $X \backslash L$ of energy $< E$ is bounded by $(M_E + D_E)/2$.*
□

Since the first Chern class of the normal bundle $N(Y, X)$ is represented by $D\omega|_Y$, the adjunction formula

$$c_1(TY) = c_1(TX|_Y) - c_1\big(N(Y, X)\big) = c_1(TX|_Y) - D[\omega|_Y]$$

shows that raising the degree D of Y lowers the expected dimension of moduli spaces of nonconstant J-holomorphic spheres in Y. So for D sufficiently large and energy bounded by E, these do not occur (because $J|_Y$ is regular on simple curves). Next, observe that the intersection number with Y of a nonconstant punctured J-holomorphic curve $f : \Sigma \to X \backslash L$ satisfies

$$[f] \cdot [Y] = D \int_\Sigma f^*\omega \geq D/D_0$$

because $D_0[\omega] \in H_2(X, L; \mathbb{Z})$. So the intersection number is positive and increases with D. Since a point of tangency of order ℓ to Y contributes $(\ell + 1)$ to the intersection number (see [20, Proposition 7.1]), and the maximal order of tangency for holomorphic planes is bounded by Lemma 9.2, we have shown

Lemma 9.3 *For D sufficiently large (depending only on ω, J_0 and E) and J^+ as above the following holds:*

(i) all J^+-holomorphic spheres of energy $< E$ in Y are constant;
(ii) all nonconstant punctured J^+-holomorphic curves in $X \backslash L$ intersect Y;
(iii) all nonconstant J^+-holomorphic planes of energy $< E$ in $X \backslash L$ intersect Y in at least 2 distinct points in the domain. □

Now consider a homology class $A \in H_2(X; \mathbb{Z})$ of holomorphic spheres we want to split along M, where $\omega(A) < E$ [A will be the class of a complex line in $\mathbb{CP}^n$ in the case of Theorem 1.2(b)]. Note that the intersection number of J-holomorphic spheres in the class A with the hypersurface Y equals

$$\ell := [Y] \cdot A = D\,\omega(A).$$

Consider the space $\mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*)$ of coherent perturbations in the bundle (TX^*, J) over the split manifold $X^* = X^+ \amalg (\mathbb{R} \times M) \amalg X^-$ as in Definition 6.4 (consisting of Floer's C^ε-sections). Let

$$\mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x) \subset \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*)$$

be the subset of K which vanish near Y as well as near the point x. Pick such a K satisfying $\|K\| < 1$. Let $\overline{\mathcal{M}}^{A,K}_{1+\ell}(X^*; x, Z, Y)$ be the moduli space of stable broken K-holomorphic spheres in X^* in the class A with ℓ marked points mapped to Y, and 1 marked point passing through x and tangent of order $n-1$ to Z at x. This space is defined as the subset of the union over all $(1+\ell)$-labelled trees T and asymptotics Γ^* of the spaces $\mathcal{M}_T^{A,K}(X^*, \Gamma^*)$ defined in Sect. 7 in the paragraph on moduli spaces (with $k = \ell + 1$) cut out by the additional constraints given at z and Z. Let

$$\mathcal{M} := \mathcal{M}_{1+\ell}^{A,K}(X^*; x, Z, Y)$$

be the corresponding moduli space of unbroken K-holomorphic spheres and

$$\overline{\mathcal{M}} \subset \overline{\mathcal{M}}_{1+\ell}^{A,K}(X^*; x, Z, Y) \tag{23}$$

be its closure in $\overline{\mathcal{M}}_{1+\ell}^{A,K}(X^*; x, Z, Y)$. Note that $\mathcal{M}$ has the same expected dimension as the corresponding moduli space without the ℓ additional marked points mapped to Y. By an argument using positivity of intersections as in [20, Proposition 9.5], Lemma 9.3 implies

Proposition 9.4 *For D sufficiently large (depending only on ω, J_0 and E), (J, K) as above, and any broken holomorphic curve $f \in \overline{\mathcal{M}}$ the following holds:*

(i) all components of f contained in Y are constant;
(ii) all components of f in $X \setminus L$ have a stable domain. □

Now we specialize to the case $L = T^n$, $X = \mathbb{C}P^n$ and the homology class $A = [\mathbb{C}P^1]$. Then the moduli space $\mathcal{M}$ has expected dimension zero. According to [20, Theorem 1.2], for generic K there exists a finite signed count $\#\mathcal{M} \in \mathbb{Z}$, and according to [20, Theorem 1.3] the number $\#\mathcal{M}/\ell!$ is independent of the choice of (J, K, Y). (In [20] the perturbation K is a coherent almost complex structure, but the proofs work exactly the same for general coherent perturbations which have the form considered in [20] near Y, see [36].) Since by Proposition 3.4 the count $\#\mathcal{M}$ is nonzero for $K = 0$ and $Y = \emptyset$, it follows that $\mathcal{M}$ is nonempty for every choice of (J, K, Y). Hence in the proof of Theorem 1.2(b) we can choose the maps f_k in the moduli spaces $\mathcal{M}$ above with respect to almost complex structures J_k realizing the neck stretching, a suitable hypersurface Y, and a coherent perturbation K in $\mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x)$.

Now we can conclude the transversality part of the proof of Theorem 1.2(b). In view of the preceding discussion, it suffices to show that all the components of broken holomorphic spheres in $\overline{\mathcal{M}}$ belong to moduli spaces that are manifolds of the expected dimensions, and the asymptotic evaluation maps are transverse to the diagonal.

Proof of transversality for Theorem 1.2(b) Since Theorem 1.2(b) is trivial for $n = 1$, let us assume $n \geq 2$. Now L is an n-dimensional torus with a multiple of the standard flat metric. Hence there are no holomorphic planes in T^*L and $\mathbb{R}\times M$, and the only holomorphic cylinders in T^*L and $\mathbb{R}\times M$ are orbit cylinders in $\mathbb{R}\times M$, and holomorphic cylinders in T^*L with two positive punctures. According to Proposition 9.4, all other punctured holomorphic spheres in T^*L, $\mathbb{R}\times M$ or $X\backslash L$ of energy $< E$ appearing in a curve in $\overline{\mathcal{M}}$ have a stable domain.

Consider $K \in \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x)$ and a stable K-holomorphic map $(\mathbf{z}, \mathbf{f}) \in \overline{\mathcal{M}}$. Thus $\mathbf{z}$ is a nodal curve modelled over a (not necessarily stable) $(1+\ell)$-labelled tree T such that the map $f_\alpha : \dot{S}_\alpha \to X^*$ is nonconstant over each unstable component $\alpha \in T$. As in Sect. 7, we consider the linearization L of

$$\begin{aligned}\bar\partial \times \mathrm{Ev} : \mathcal{M}_T \times \mathcal{B}^s_T(X^*, \Gamma^*) \times \mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x)\\ \to \mathcal{E} \times \prod_{\alpha E\beta} \Gamma_{\alpha\beta} \times \prod_{1\le i\le 1+\ell} \Gamma_i\end{aligned}$$

at $(\mathbf{f}, K)$. Here the normal $(n-1)$-jet evaluation map at z_1 takes values in $\Gamma_1 = (T_xT^*L/T_xZ)^n$, while the evaluation maps at the points z_i take values in $\Gamma_i = X\backslash L$ for $i = 2, \dots, \ell+1$. The points $z_{\alpha\beta}$ can correspond to punctures, in which case $\Gamma_{\alpha\beta}$ is a component of the space of closed geodesics, or to nodes, in which case $\Gamma_{\alpha\beta}$ is a component of X^*. In contrast to Proposition 7.4, we cannot conclude that L is surjective for two reasons:

(1) The perturbations in $\mathcal{K}^\varepsilon(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x)$ are required to vanish near Y and near x, so they are of no use to achieve transversality of the evaluation map on components f_α that are mapping entirely into $Y \cup \{x\}$. This can be remedied as follows.

Note first that no component can be mapped entirely into the point x. Indeed, such a component would be constant and carry at most one marked point (because the points $z_2, \dots, z_{\ell+1}$ are mapped to Y and $x \notin Y$). In view of stability, the component must carry at least two nodal points and thus split the tree into at least two subtrees. Each of these subtrees must contain a component in $\mathbb{C}P^n\backslash L$ and thus represent a nontrivial homology class in $\mathbb{C}P^n$, which is impossible because the total homology class $[\mathbb{CP}^1]$ is indecomposable.

Next consider a component f_α that is mapped entirely into Y. By Proposition 9.4(a), the map f_α is constant. Following [20], we denote by $T_1 \subset T$ the maximal subtree containing α consisting of constant components in Y (a "ghost tree"). Indecomposability of the homology class $[\mathbb{CP}^1]$ implies that $T\backslash T_1$ is connected, so by stability T_1 must contain at least two of the marked points $z_2, \dots, z_{\ell+1}$. According to [20, Proposition 7.1 and Lemma 7.2], the nonconstant map f_β adjacent to T_1 is tangent to Y at the point $z_{\beta\alpha}$. We replace T by $T\backslash T_1$ and consider the remaining moduli space with an additional tangency condition to Y imposed at the new marked point $z_{\beta\alpha}$. The new evaluation map (still denoted by Ev) contains the 1-jet evaluation map at $z_{\alpha\beta}$. This modification is performed for all ghost trees that are mapped to Y. Note that this decreases the number of marked points mapped to Y, but we keep denoting it by ℓ.

(2) The tree T need not be stable, and by construction the perturbations in $\mathcal{K}^{\varepsilon}(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x)$ are domain independent on unstable components. Now by the discussion above, unstable components can only correspond to cylinders in T^*L with two positive punctures (orbit cylinders can be removed from a stable map). Thus no two such components can be adjacent. Moreover, by Lemma 7.3, they appear in regular moduli spaces of the expected dimension $2(n-1)$ whose image under the evaluation map is a covering of T^*L. So we can use surjectivity of the linearized evaluation map at the adjacent components to achieve transversality to the diagonal at the punctures of each such component.

In view of this discussion and Proposition 7.4 (with $k = 1+\ell$), after the modification in (1) the linearized operator L is transverse to the manifold

$$\mathcal{Z}_T := \{0_{\mathcal{E}}\} \times \prod_{\alpha E \beta} \Delta_{\alpha\beta} \times \{0_{\Gamma_1}\} \times \prod_{2\le i\le 1+\ell} Y_i.$$

Here $\Delta_{\alpha\beta}$ denotes the diagonal in $\Gamma_{\alpha\beta}\times\Gamma_{\beta\alpha}$ at the edge $\alpha E\beta$, and Y_i denotes TY at the new marked points z_i where we take the 1-jet evaluation map, and Y at the other ones.

Now the proof is finished by the standard argument as in the proof of Corollary 7.6: For each tree T as above, the implicit function theorem implies that $\mathcal{U}^s_T = (\bar\partial \times \mathrm{Ev})^{-1}(\mathcal{Z}_T)$ is a Banach manifold. Pick $K \in \mathcal{K}^{\varepsilon,*}(\overline{\mathcal{M}}_{1+\ell+1}, TX^*; Y, x)$ to be a regular value of the projection onto the third factor of $\mathcal{U}^s_T$ for each tree T, as well as for all spaces corresponding to subtrees of T (this exists by the Sard-Smale theorem). Then all components of broken holomorphic spheres in the corresponding moduli space $\overline{\mathcal{M}}$ defined in (23) appear in manifolds of the expected dimensions, and the evaluation maps at the edges are transverse to the diagonal. Hence for each tree T the corresponding moduli space $\mathcal{M}_T \subset \overline{\mathcal{M}}$ (taking preimages of the diagonals under the edge evaluation maps) is a smooth manifold. According to Corollary 7.6, its dimension is given by

$$\begin{aligned}\dim \mathcal{M}_T &= (2n-6) + (2n+2) - (4n-4) + 2\ell - \sum_{i=2}^{\ell+1} \mathrm{codim}(Y_i) - 2m(T)\\ &= 2\ell - \sum_{i=2}^{\ell+1} \mathrm{codim}(Y_i) - 2m(T).\end{aligned}$$

Here the term $(2n+2)$ comes from $c_1(\mathbb{CP}^n)$ evaluated on $[\mathbb{C}P^1]$, $(4n-4)$ from the tangency condition to Z at x (see Sect. 5), $\mathrm{codim}(Y_i)$ equals 2 if

$Y_i = Y$ and 4 if $Y_i = TY$, and $m(T)$ is the number of nodes (i.e., edges of T over which the maps extend continuously). It follows that the moduli space can be nonempty (and thus the dimension nonnegative) only if $m(T) = 0$ and $Y_i = Y$ for all i, so nodes and components mapped into Y as discussed in (1) above cannot occur. (Nodes are also excluded by indecomposability of the homology class $[\mathbb{CP}^1]$.)

To sum things up, we have shown that all moduli spaces of punctured K-holomorphic spheres corresponding to the vertices of a tree associated to a limit of a sequence of K_k-holomorphic spheres passing through x the with jet constraint to Z are transversely cut out and the pairs of asymptotic evaluation maps corresponding to an edge are transverse to the diagonal. By construction, the same is true for the moduli spaces and their asymptotic evaluation maps for any (connected) subtree.

Recall now the definition of the moduli spaces $\mathcal{M}$ and $\mathcal{M}_i$ for $i = 1, \ldots, m$ from Sect. 5 associated to the limit curve [here we change to the notation in Sect. 5, so $\mathcal{M}$ is no longer the space appearing in (23)]: Let α_1 be the vertex of the corresponding tree T containing the marked point z_1 which is mapped to x, and let $T_1, \ldots, T_m$ be the subtrees obtained by removing this vertex from T. Then $\mathcal{M}$ is the moduli space of punctured K-holomorphic spheres with m positive punctures in T^*L corresponding to the vertex α, and $\mathcal{M}_i$ is the moduli space of broken K-holomorphic spheres with one negative puncture corresponding to the subtree T_i. Again by Corollary 7.6, the dimensions of these moduli spaces are given by

$$\dim \mathcal{M}_i = (n-3) - \mathrm{CZ}(\Gamma_i), \qquad i = 1, \ldots, m,$$
$$\dim \mathcal{M} = (n-3)(2-m) + (2n+2) + \sum_{\imath=1}^{m} \big(\mathrm{CZ}(\Gamma_i) + \dim \Gamma_i\big) - (4n-4)$$

and the asymptotic evaluation maps of $\mathcal{M}_i$ are transverse to that of $\mathcal{M}$. This concludes the proof of transversality for Theorem 1.2(b). □

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A The Chekanov torus

In this section we prove the "No Go" result mentioned in the Introduction: *For the Chekanov torus* $T_{\mathrm{Chekanov}} \subset \mathbb{CP}^2$, *the boundary loops of the three disks produced in Theorem* 1.1 *will in general represent multiples of the same* *homology class and thus fail to generate the first homology of the torus.* More precisely, we will see that this occurs whenever we choose the almost complex structure in the proof of Theorem 1.1 from a suitable nonempty open set.

We begin by recalling the construction of the Chekanov torus $T_{\text{Chekanov}} \subset \mathbb{CP}^2$ following [16]. Let $\gamma \subset \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ be a closed embedded curve in the right half-plane, bounding a closed disk D_γ of area $\pi/3$ which contains $1 \in \mathbb{C}$ in its interior. Denote their images under the diagonal embedding $z \in \mathbb{C} \mapsto [1 : z : z] \in \mathbb{CP}^2$ by

$$\Gamma := \{[1 : z : z] \mid z \in \gamma\} \subset D_\Gamma := \{[1 : z : z] \mid z \in D_\gamma\} \subset \mathbb{CP}^2.$$

Consider the symplectic S^1-action on $\mathbb{CP}^2$ given by $e^{i\theta} \cdot [z_0 : z_1 : z_2] := [z_0 : e^{i\theta} z_1 : e^{-i\theta} z_2]$. The *Chekanov torus* is the union of the orbits of this action through Γ,

$$L := T_{\text{Chekanov}} := \{[1 : e^{i\theta} z : e^{-i\theta} z] \mid z \in \gamma,\ e^{i\theta} \in S^1\} \subset \mathbb{CP}^2.$$

Note that $\partial D_\Gamma = \Gamma \subset L$. We pick a point $v \in \gamma$ with $v \notin \mathbb{R}$ and set

$$\tau := \{[1 : e^{i\theta} v : e^{-i\theta} v] \mid e^{i\theta} \in S^1\} \subset D_\tau := \{[1 : \zeta v : \bar{\zeta} v] \mid \zeta \in D\} \subset \mathbb{CP}^2,$$

where $D \subset \mathbb{C}$ denotes the closed unit disk. Thus τ is the orbit of the point $[1 : v : v]$ under the S^1-action and $\partial D_\tau = \tau \subset L$. We equip D_Γ and D_τ with the orientations induced from D_γ or Δ, respectively, via the embeddings. Finally, let us introduce the projective lines S_0, S_1, S_2 and the quadric Q in $\mathbb{CP}^2$,

$$S_i := \{[z_0 : z_1 : z_2] \mid z_i = 0\},\ i = 0, 1, 2, \qquad Q := \{[z_0 : z_1 : z_2] \mid z_0^2 = z_1 z_2\}.$$

It follows directly from the defining equations that $S_0, S_1, S_2, Q \subset \mathbb{CP}^2 \setminus L$. Since $H_1(\mathbb{CP}^2) = 0$ and $[L] = 0 \in H_2(\mathbb{CP}^2)$, the long exact sequence of the pair $(\mathbb{CP}^2, L)$ shows that $H_2(\mathbb{CP}^2, L) \cong \mathbb{Z}^3$ is generated by the relative homology classes represented by D_Γ, D_τ and S_0 (which we will denote by the same symbols).

Lemma A.1 (Chekanov–Schlenk [16])

(a) The Chekanov torus $L \subset \mathbb{CP}^2$ is a monotone Lagrangian 2-torus in $\mathbb{CP}^2$ with $\int_A \omega = \frac{\pi}{6}\mu(A)$ for the Maslov class μ and all $A \in H_2(\mathbb{CP}^2, L)$.

(b) Let J be an almost complex structure on $\mathbb{CP}^2 \setminus L$ with a negative cylindrical end for which S_0, S_1, S_2 and Q are complex submanifolds. Then the only relative homology classes in $H_2(\mathbb{CP}^2, L)$ which may contain punctured J-holomorphic spheres of Maslov number 2 are

$$D_\Gamma,\ S_0 - 2D_\Gamma,\ S_0 - 2D_\Gamma \pm D_\tau.$$

Apart from the doubles of these classes, the only classes which may contain punctured J-holomorphic spheres of Maslov number 4 are

$$S_0 - D_\Gamma,\ S_0 - D_\Gamma \pm D_\tau,\ 2S_0 - 4D_\Gamma \pm D_\tau.$$

Proof Part (a) is proved in [16]. Part (b) is carried out in [16] for $S^2 \times S^2$ and the arguments easily carry over to $\mathbb{CP}^2$; the $\mathbb{CP}^2$ case is also treated in [5]. For convenience, we include the argument. The following table shows the relevant intersection and Maslov numbers in $\mathbb{CP}^2$. Here the Maslov numbers can be taken from [16] since they are the same in $S^2 \times S^2$ and $\mathbb{CP}^2$, and the intersection numbers follow easily from the defining equations and inspection of orientations at intersection points.

	D_Γ	D_τ	S_0	$A = a_\Gamma D_\Gamma + a_\tau D_\tau + bS_0$
S_0	0	0	1	$b \geq 0$
S_1	0	-1	1	$-a_\tau + b \geq 0$
S_2	0	1	1	$a_\tau + b \geq 0$
Q	1	0	2	$a_\Gamma + 2b \geq 0$
μ	2	0	6	$2a_\Gamma + 6b = 2$ or 4

Now suppose that a class $A = a_\Gamma D_\Gamma + a_\tau D_\tau + bS_0 \in H_2(\mathbb{CP}^2, L)$ contains a punctured J-holomorphic sphere. Since S_0, S_1, S_2 and Q are J-complex submanifolds, all the intersection numbers in the last column of the table must be nonnegative. Together with the condition on the Maslov number this yields the result. □

Corollary A.2 *Let $L = T_{\text{Chekanov}} \subset \mathbb{CP}^2$ be the Chekanov torus equipped with a flat metric. Let $C_0, \ldots, C_k \subset \mathbb{CP}^2 \setminus L$ be the punctured J-holomorphic spheres resulting from degenerating holomorphic spheres in the class* $[\mathbb{CP}^1]$ *along the boundary of a tubular neighbourhood of L, where J is an almost complex structure on $\mathbb{CP}^2 \setminus L$ satisfying the conditions in Lemma* A.1(b). *Then the boundary loops of the C_i all represent multiples of the same homology class* $[\Gamma] \in H_1(L)$.

Proof Let $F = (F^{(1)}, \ldots, F^{(N)})$ be a broken holomorphic sphere in $\mathbb{CP}^2$ with N obtained as the limit of holomorphic spheres in the class $[\mathbb{CP}^1]$ in the neck-stretching procedure. Thus the components $C_0, \ldots, C_k$ of $F^{(N)}$ are punctured J-holomorphic spheres in $\mathbb{CP}^2 \setminus L$, where we assume that the almost complex structure J on $\mathbb{CP}^2 \setminus L$ satisfies the conditions in Lemma A.1(b). Note that all components C_i have positive symplectic area and the sum of the areas equals π. By monotonicity of L, the Maslov number of each C_i equals $6/\pi$ times its area. So all Maslov numbers are positive even integers and their sum equals 6, which leaves only the combinations (2, 2, 2), (4, 2) and (2, 4).

Since the total intersection number of F with S_0 equals 1, there is a unique component, say C_0, which intersects S_0. According to Lemma A.1, the components which do not intersect S_0 are homologous to either D_Γ or $2D_\Gamma$ in $H_2(\mathbb{CP}^2, L)$. Since the sum of the relative homology classes of all the components of $F^{(N)}$ equals S_0, this leaves only the following possibilities for the relative homology classes of $C_0, \dots, C_k$:

$$(S_0 - 2D_\Gamma, D_\Gamma, D_\Gamma) \quad \text{or} \quad (S_0 - 2D_\Gamma, 2D_\Gamma) \quad \text{or} \quad (S_0 - D_\Gamma, D_\Gamma).$$

In particular, the boundary of each component C_i is homologous to a multiple of Γ (where the boundary of C_i is the sum of the asymptotic geodesics at its punctures). Hence either all asymptotic geodesics of such a component are homologous to multiples of Γ, or at least two of them are not. The same holds for the components of the broken holomorphic curve F in T^*L and $\mathbb{R} \times S^*L$. Recall that one may assign to F a tree whose nodes correspond to the components of F, and whose edges correspond to common asymptotic geodesics between two nodes resulting from the limit process. Hence the ends of the tree correspond to components with exactly one puncture which, by the preceding discussion, has to be asymptotic to a multiple of Γ. Remove all ends together with their adjacent edges. For each punctured holomorphic sphere corresponding to a node of the remaining tree it is still true that the asymptotics of the punctures corresponding to the remaining nodes are either all homologous to multiples of Γ, or at least two of them are not. (This holds since all edges we removed correspond to punctures asymptotic to multiples of Γ.) By induction over the number of nodes in the tree (since it holds for one node), it follows that all asymptotic geodesics are homologous to multiples of Γ and the corollary follows. □

Remark A.3 (a) For the Chekanov torus, the proof of Theorem 1.1 (degenerating holomorphic spheres with a tangency condition) produces precisely three holomorphic planes C_0, C_1, C_2 in $\mathbb{CP}^2 \backslash L$, which by the proof of Corollary A.2 must represent the relative homology classes $(S_0 - 2D_\Gamma, D_\Gamma, D_\Gamma)$.

(b) Corollary A.2 continuous to hold (with the same proof) for a tamed almost complex structure J for which there exist complex submanifolds in $\mathbb{CP}^2 \backslash L$ that are homologous (in the complement of L to S_0, S_1, S_2 and Q. The set of these almost complex structures is open.

(c) Carrying over arguments from [16] to the $\mathbb{CP}^2$ case, it is shown in [5] that each of the relative homology classes of Maslov number 2 listed in Lemma A.1 contains a unique holomorphic disk passing through a given point on L. Neck stretching at the boundary of a tubular neighbourhood of L yields holomorphic planes in $\mathbb{CP}^2 \setminus L$ representing these classes. This shows that there exist collections of holomorphic planes, e.g. three planes representing the classes $(S_0 - 2D_\Gamma + D_\tau, S_0 - 2D_\Gamma - D_\tau, 4D_\gamma)$, whose asymptotic geodesics do generate $H_1(L)$. According to Corollary A.2, such collections of planes do not arise in degenerations of holomorphic spheres in the class $[\mathbb{CP}^1]$.

B A non-removable intersection

Here we prove the claim in Remark 1.10 from the introduction, which we restate as follows.

Proposition B.1 *Let $(L_t)_{t\in[0,1]}$ be a Hamiltonian isotopy of closed Lagrangian submanifolds in the closed unit ball $B \subset \mathbb{C}^n$. If $L_0 \subset \partial B$, then $L_t \subset \partial B$ for all $t \in [0, 1]$.*

Proof The set $\{t \in [0, 1] \mid L_t \subset \partial B\}$ is clearly nonempty and closed, so we only need to show it is open. So let $L := L_t \subset \partial B$. Let v be the vector field on ∂B generating the Hopf fibration (which is the characteristic foliation). Then $\omega(v, w) = 0$ for all $w \in TL \subset T(\partial B)$, so $v \in (TL)^{\perp_\omega} = TL$. This shows that L is invariant under the Hopf fibration, hence it descends to a Lagrangian submanifold $\bar{L} \subset \mathbb{C}P^{n-1}$. By the Lagrangian neighbourhood theorem and symplectic reduction, a neighbourhood of L in $\mathbb{C}^n$ is symplectomorphic to a neighbourhood U of the zero section in

$$(T^*\bar{L} \times T^*S^1, dp \wedge dq).$$

Here $(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1})$ are canonical coordinates on $T^*\bar{L}$ and (q_n, p_n) on T^*S^1. Moreover, the variable p_n corresponds to the moment map $z \mapsto \pi(|z|^2 - 1)$ of the standard circle action on $\mathbb{C}^n$, so $p_n > 0$ outside B and $p_n \le 0$ in B.

For s sufficiently close to t we can write L_s as the graph of an exact 1-form in U, i.e.,

$$L_s = \left\{ (q, p) \mid p_i = \frac{\partial f}{\partial q_i} \right\}$$

for a function $f : \bar{L} \times S^1 \to \mathbb{R}$. Fix a point $\bar{q} = (q_1, \dots, q_{n-1})$. Since L_s is contained in B, we have $\frac{\partial f}{\partial q_n}(\bar{q}, q_n) = p_n \le 0$ for all q_n. On the other hand,

$$\int_0^{2\pi} p_n \, dq_n = \int_0^{2\pi} \frac{\partial f}{\partial q_n}(\bar{q}, q_n) dq_n = 0.$$

So p_n must vanish identically, which means that $L_s \subset \partial B$. □

C The embedding capacity for the flat torus

Let $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ be the standard flat torus, normalized such that each S^1-factor has length 2π. Equip $\mathbb{CP}^n$ with the standard symplectic form such that a complex line has area π, so $\mathrm{vol}(\mathbb{CP}^n) = \mathrm{vol}B^{2n}(1)$. Then the volume estimate (1) becomes

$$c^{\mathbb{CP}^n}(T^n) \geq \sqrt[n]{\frac{(2\pi)^n \mathrm{vol}B^n(1)}{\mathrm{vol}B^{2n}(1)}} =: C_n, \tag{24}$$

whereas the estimate from Theorem 1.22 reads

$$c^{\mathbb{CP}^n}(T^n) \geq 2(n+1). \tag{25}$$

In view of the well-known formulae

$$\mathrm{vol}B^{2k}(1) = \frac{\pi^k}{k!}, \qquad \mathrm{vol}B^{2k+1}(1) = \frac{2^{2k+1}\pi^k k!}{(2k+1)!}$$

the right hand side of Eq. (24) can be written as

$$C_{2k} = 2\sqrt{\pi}\,\sqrt[2k]{\frac{(2k)!}{k!}}, \qquad C_{2k+1} = 4\,\sqrt[2k+1]{\pi^k k!}.$$

Note that the estimates (24) and (25) agree for $n = 1$.

Lemma C.1 *For every integer $n \geq 2$ we have $C_n < 2(n+1)$, so the estimate (25) is better than the volume estimate (24).*

Proof For $n = 2k \geq 2$ even we use $2\pi k < (2k+1)^2$ to estimate

$$C_{2k} \leq 2\sqrt{\pi}\,\sqrt[2k]{(2k)^k} = 2\sqrt{2\pi k} < 2(2k+1) = 2(n+1).$$

Similarly, for $n = 2k+1 \geq 3$ odd we use $\pi k < (k+1)^2$ to estimate

$$C_{2k+1} = 4\,\sqrt[2k]{\pi^k k^k} = 4\sqrt{\pi k} < 4(k+1) = 2(n+1).$$

□

Finally, we wish to compare the estimates to the values realized by the obvious symplectic embeddings

$$\phi : \Big((-1,1)\times(\mathbb{R}/2\pi\mathbb{R})\Big)^n \hookrightarrow \mathbb{C}^n, \qquad (s_j, t_j) \mapsto z_j := \sqrt{2+2s_j}\,e^{it_j}.$$

Lemma C.2 *The embedding ϕ yields the upper bound*

$$c^{\mathbb{CP}^n}(T^n) \leq 2(n+\sqrt{n}),$$

which agrees with the lower bound (25) iff $n = 1$.

Proof In the unit co-diskbundle $D^*(T^n)$ we have $\sum_{j=1}^n s_j^2 \leq 1$. The maximum of the squared norm $|z|^2 = \sum_{j=1}^n (2+2s_j)$ on the image of ϕ under this constraint is attained for $s_1 = \cdots = s_n = 1/\sqrt{n}$ and given by

$$\max|z^2| = \sum_{j=1}^n (2+2/\sqrt{n}) = 2(n+\sqrt{n}).$$

Rescaling the symplectic form by this factor yields the upper bound. □

D Gromov–Hofer compactness for K-holomorphic maps

In this appendix we extend the Gromov–Hofer Compactness Theorem 2.9 to K-holomorphic maps.

Let (X_k, J_k, ω_k) be a sequence of symplectic manifolds with almost complex structures as in Theorem 2.9, corresponding to neck stretching at a Morse Bott contact type hypersurface $(M, \lambda) \subset (X, \omega)$ and converging to the split cobordism (X^*, J^*). Fix an integer $m \geq 3$ and a coherent perturbation $K \in \mathcal{K}(\overline{\mathcal{M}}_{m+1}, TX^*)$ in the sense of Definition 6.4 satisfying the taming condition $\|K\| \leq 1$ with respect to a fixed Kähler form σ on $\overline{\mathcal{M}}_{m+1}$. For $k \in \mathbb{N}$ we define $K_k \in \mathcal{K}(\overline{\mathcal{M}}_{m+1}, TX_k)$ by the restriction of K on $X_0^+ \cup X_0^-$ and the canonical $\mathbb{R}$-invariant extension of the element in $\mathcal{K}(\overline{\mathcal{M}}_{m+1}, \xi)$ induced by K on $[-k, 0] \times M$.

Theorem D.1 (Gromov–Hofer compactness for K-holomorphic maps) *Let (Σ_k, j_k) be a sequence of spheres with m marked points, and $f_k : (\Sigma_k, j_k) \to (X_k, J_k, K_k)$ be a sequence of K_k-holomorphic curves in the same homology class $[f_k] = A \in H_2(X; \mathbb{Z})$. After passing to a subsequence, f_k converges in the sense of Theorem 2.9 to a broken K-holomorphic map $F : (\Sigma^*, j) \to (X^*, J^*, K)$.*

D.1 Conformal modulus of annuli in hyperbolic surfaces

The proof relies on two lemmas about the conformal modulus of annuli in hyperbolic surfaces. Recall the definition of the conformal modulus from [1], see also [19]. Let (Σ, j) be a Riemann surface, possibly with smooth boundary, and let Γ be an isotopy class of simple smooth loops in Σ. For a measurable conformal metric ρ, let

$$L_\rho(\Gamma) := \inf_{\gamma\in\Gamma} L_\rho(\gamma).$$

be the infimum of the ρ-length $L_\rho(\gamma)$ of curves $\gamma \in \Gamma$. Define the *modulus of* Γ by

$$M(\Gamma) := \inf_\rho \frac{\mathrm{area}_\rho(\Sigma)}{L_\rho(\Gamma)^2},$$

where the infimum is taken over all measurable conformal metrics with $0 < L_\rho(\Gamma) < \infty$. Clearly, $M(\Gamma)$ is a conformal invariant. Its reciprocal, $1/M(\Gamma)$, is known as the "extremal length". The *modulus of an annulus* A is defined as $\mathrm{Mod}(A) := M(\Gamma_A)$, where Γ_A is the class of simple loops isotopic to a boundary component. Then the annulus $[0, L] \times \mathbb{R}/\mathbb{Z}$ with the standard complex structure has modulus L.

Lemma D.2 (Maskit [53]) *Let (Σ, h) be a complete hyperbolic surface (possibly with boundary). Let $\gamma \subset \Sigma$ be a simple closed geodesic which is contained in an embedded annulus $A \subset \Sigma$. Then*

$$L_h(\gamma) \le \frac{2\pi}{\mathrm{Mod}(A)}.$$

Proof Let Γ be the free homotopy class of γ in Σ. Then the lemma follows from Maskit's inequality $L_h(\gamma) \le 2\pi/M(\Gamma)$ (see [53,54] where the inequality is stated in terms of extremal length) and the obvious inequality $\mathrm{Mod}(A) \le M(\Gamma)$. □

Let (Σ, h) be a complete hyperbolic surface (without boundary). For $0 < \delta < \mathrm{arsinh}(1)$ define the *δ-thin part* of (Σ, h) as the set of all points at which the injectivity radius is less than δ. It is proved in [45] that every component of the δ-thin part is either a punctured disk around a puncture or an annulus around a closed geodesic.

Lemma D.3 *For a complete hyperbolic surface (Σ, h) of Euler characteristic $\chi(\Sigma)$ and $0 < \delta < \mathrm{arsinh}(1)$ set $\ell := -2\pi\chi(\Sigma)/\delta^2$. Then for every conformal embedding $\phi : [-L, L] \times \mathbb{R}/Z \hookrightarrow \Sigma$ of an annulus of modulus $L > \ell$ such that $\phi(\{0\} \times \mathbb{R}/\mathbb{Z})$ is noncontractible in Σ, the smaller annulus $\phi([-L + \ell, L - \ell] \times \mathbb{R}/Z)$ is contained in the δ-thin part of (Σ, h).*

Proof Consider the annulus $A := \phi([-L, -L + \ell] \times \mathbb{R}/Z) \subset \Sigma$. Then

$$\ell = \mathrm{Mod}(A) = \inf_\rho \frac{\mathrm{area}_\rho(A)}{L_\rho(\Gamma_A)^2} \le \frac{\mathrm{area}_h(A)}{L_h(\Gamma_A)^2} \le \frac{\mathrm{area}_h(\Sigma)}{L_h(\Gamma_A)^2} = \frac{-2\pi\chi(\Sigma)}{L_h(\Gamma_A)^2},$$

where the last equality follows from the Gauss–Bonnet theorem. By definition of ℓ this becomes $L_h(\Gamma_A) \leq \delta$, so there exists a simple loop $\gamma \subset A$ isotopic to a boundary component of length $L_h(\gamma) < 2\delta$. The noncontractibility hypothesis on ϕ implies that γ is noncontractible in Σ. By definition of the injectivity radius, γ must therefore be entirely contained in the δ-thin part of (Σ, h).

The same argument applied to the annulus $A' := \phi([L-\ell, -L] \times \mathbb{R}/Z)$ yields a simple loop $\gamma' \subset A'$ isotopic to a boundary component which is noncontractible in Σ and contained in the δ-thin part of (Σ, h). Since different components of the δ-thin part have non-isotopic boundary loops, γ and γ' must belong to the same component. It follows that the annulus $B \subset A$ bounded by γ and γ', and therefore also its subset $\phi([-L+\ell, L-\ell] \times \mathbb{R}/Z) \subset B$, is entirely contained in the δ-thin part. □

D.2 Gromov compactness with free boundary

The second ingredient in the proof is the extension of the Gromov compactness theorem with free boundary [19, Theorem 3.2] to K-holomorphic maps. Although Gromov compactness for K-holomorphic maps has been considered at various places in the literature, the references we are aware of such as [37,56] do not treat the case at hand in which the underlying domains on which the K depends can pinch to nodal curves. So we include the proof for the sake of completeness.

Let (X, J, ω) be a (not necessarily compact) almost complex manifold with a compatible nondegenerate (not necessarily closed) 2-form ω, or equivalently, with a Hermitian metric $\omega(\cdot, J\cdot)$. Following [19], we impose the following conditions on a map $f : E \to X$ defined on a subset $E \subset \Sigma$:

- **(A1)** (Σ, j, σ) is a closed Riemann surface of genus g with m distinct marked points $\mathbf{z} = (z^1, \ldots, z^m)$ and a positive area form σ, and $E \subset \Sigma$ is a compact connected subsurface with boundary.
- **(A2)** The energy of f satisfies $E(\widehat{f}) := \int_E \sigma + \int_E f^*\omega \leq C$ for a constant $C > 0$.
- **(A3)** The image of f is contained in a compact subset $B \subset X$.
- **(A4)** At the boundary components γ of (E, j) there exist mutually disjoint conformal embeddings $\beta^\gamma : [0, 5L] \times \mathbb{R}/\mathbb{Z} \hookrightarrow E \backslash \{z^1, \ldots, z^m\}$ mapping $\{0\} \times \mathbb{R}/\mathbb{Z}$ onto γ for some $L \geq 1$.
- **(A5)** For each boundary component γ of E, the differential of $f \circ \beta^\gamma$ satisfies $1/D \leq \|T(f \circ \beta^\gamma)\| \leq D$ with respect to the Euclidean metric on $[0, 5L] \times \mathbb{R}/\mathbb{Z}$, for some constant $D > 0$.

Remark D.4 The condition $L \geq 1$ in (A4) replaces the condition $L \geq L_0$ in [19], where L_0 depended on the injectivity radius and the constants in the monotonicity lemma and Schwarz lemma on the target space. Such a dependence would be problematic in the following proof where the target space $\Sigma_k \times X$ depends on k. Lemma D.2 allows us to avoid this dependence and use the uniform condition (A4).

Proposition D.5 (Gromov compactness with free boundary for K-holomorphic maps) *Let (X, J, ω) be an almost complex manifold with compatible nondegenerate 2-form ω. Let $(\Sigma_k, j_k, \mathbf{z}_k, \sigma_k)$, $E_k \subset \Sigma_k$ and $f_k : E_k \to X$ satisfy conditions (A1-5) with g, m, C, B, L, D independent of $k \in \mathbb{N}$. Suppose that the f_k are K_k-holomorphic where $K_k \in \mathcal{K}(\dot{\Sigma}_k, TX)$ are perturbations as defined in (10), satisfying the taming condition $\|K_k\| \le 1$ with respect to σ_k and vanishing on the δ-thin part of $\dot{\Sigma}_k = \Sigma_k \setminus \{z_1, \dots, z_m\}$. Suppose that $(\Sigma_k, j_k, \mathbf{z}_k, \sigma_k)$ converges to a nodal surface $(\overline{\Sigma}, j, \mathbf{z}, \sigma)$ and K_k converges to $K \in \mathcal{K}(\Sigma^*, TX)$, where Σ^* is the complement of the marked points and nodes in $\overline{\Sigma}$. Then a subsequence of (f_k) converges in the sense of [19, Definition 3.1] to a nodal K-holomorphic curve $f : (E, j_E) \to (X, J, K)$ together with a holomorphic map $(E, j_E) \to (\overline{\Sigma}, j)$.*

Proof Recall the definitions of the almost complex structure $\widehat{J}_k$ from (13) and the symplectic structure $\widehat{\omega}_k$ from (12) on $\Sigma_k \times X$. We consider the graphs $\widehat{f}_k = \mathrm{id} \times f_k : (E_k, j_k) \to (\Sigma_k \times X, \widehat{J}_k, \widehat{\omega}_k)$ as ordinary holomorphic maps. We identify each Σ_k with a fixed model surface Σ such that $j_k \to j$ in C^∞_{loc} and $\mathbf{z}_k \to \mathbf{z}_\infty$ on the complement $\Sigma^* = \Sigma \setminus \Delta$ of a finite union $\Delta \subset \Sigma$ of disjoint embedded loops. Pick a tubular neighbourhood $[-\varepsilon, \varepsilon] \times \Delta$ of Δ (corresponding to the δ-thin part) on which K_k vanishes for large k. For each component Δ^i of Δ we have biholomorphisms $([-\varepsilon, \varepsilon] \times \Delta^i, j_k) \cong ([-c^i_k - \varepsilon, \varepsilon] \times S^1, i)$ where $S^1 = \mathbb{R}/\mathbb{Z}$, i is the standard complex structure, and $c^i_k \to \infty$ as $k \to \infty$. Thus

$$\Big([-\varepsilon, \varepsilon] \times \Delta^i \times X, \widehat{J}_k\Big) \cong \Big([-c^i_k - \varepsilon, \varepsilon] \times S^1 \times X, i \oplus J\Big),$$

so the almost complex structures $\widehat{J}_k$ on the target manifold $\Sigma \times X$ exhibit neck stretching at the hypersurface $M := \Delta \times X$ and converge in C^∞_{loc} on $(\Sigma \times X) \setminus M$. Note that the (M, ω, dt) is a stable Hamiltonian hypersurface of Morse–Bott type, where t is the coordinate on S^1.

So we are in the situation of the Gromov–Hofer Compactness Theorem 2.9 for ordinary J-holomorphic maps with 4 adjustments: The almost complex structure on the target is k-dependent and converges away from the hypersurface M; the inserted cylindrical piece $[-k, 0] \times M$ is replaced by a union $[-c^i_k, 0] \times M_i$ over the components M_i of M; the manifold X is noncompact; and the domains E_k have boundary. The first two adjustments are purely notational, and the noncompactness of X causes no trouble because all the images are contained in a fixed compact subset. The boundary of E_k is dealt with by combining the proof of Theorem 2.9 with the proof of the Gromov compactness theorem with free boundary for J-holomorphic maps [19, Theorem 3.2], using assumptions **(A4)** and **(A5)** near the boundary. Here replacing Lemma 3.10 in [19] by Lemma D.2 above allows us to replace in (A4) the condition $L \geq L_0$ from [19] by the condition $L \geq 1$.

It follows that a subsequence of (f_k) converges to a broken punctured holomorphic map $F : (E^*, j_E) \to (\Sigma^* \times X, \widehat{J}^*)$. The first component of F extends continuously over the punctures to a holomorphic map $(E, j_E) \to (\overline{\Sigma}, j)$ which collapses some spheres to points. The second component of F extends continuously over the punctures to a nodal K-holomorphic map $f : (E, j_E) \to (X, J, K)$. □

Proof of Theorem D.1 Let $f_k : (\Sigma_k, j_k) \to (X_k, J_k, K_k)$ be as in the theorem. Let σ_k be the area form on Σ_k obtained by restricting the fixed Kähler form σ to the corresponding fibre of $\overline{\mathcal{M}}_{m+1} \to \overline{\mathcal{M}}_m$. By Stokes' theorem, the total area $\int_{\Sigma_k} \sigma_k$ is independent of k, say equal to a.

Recall that the graph $\widehat{f_k} := \mathrm{id} \times f_k : \Sigma_k \to \Sigma_k \times X_k$ is $\widehat{J_k}$-holomorphic for the almost complex structure $\widehat{J_k}$ on $\Sigma_k \times X_k$ determined by J_k and K_k. Since $\widehat{J_k}$ is tamed by $\widehat{\omega}_k = \sigma_k \oplus \omega_k$, the integrand $\widehat{f_k}^* \widehat{\omega}_k$ is nonnegative. The hypothesis $[f_k] = A \in H_2(X; \mathbb{Z})$ and the equality of cohomology classes $[\omega_k] = [\omega]$ yields the uniform (i.e., k-independent) energy bound

$$E(\widehat{f_k}) := \int_{\Sigma_k} \widehat{f_k}^* \widehat{\omega}_k = \int_{\Sigma_k} \sigma_k + \int_{\sigma_k} f_k^* \omega_k = a + \int_A \omega =: E_0.$$

Recall that by definition the coherent perturbation K vanishes on the δ-thin part of every fibre of the projection $\overline{\mathcal{M}}^*_{m+1} \to \overline{\mathcal{M}}_m$ (where $\overline{\mathcal{M}}^*_{m+1} \subset \overline{\mathcal{M}}_{m+1}$ is the complement of the special points) with respect to its unique complete hyperbolic metric. It follows that each K_k vanishes on the δ-thin part of (Σ_k, j_k) with respect to its unique complete hyperbolic metric on the complement of the marked points.

After these preparations, the proof proceeds as in [19] with minor modifications that we will point out now. We will use the notation from [19] and consistently endow objects defined via the graph construction in $\Sigma_k \times X_k$ with a hat.

The results in [19, Section 4] on cylinders of small area are used only for *ordinary J-holomorphic maps*, i.e., without inhomogeneous perturbation K. Let $\beta_0 > 0$ be the constant defined right after Corollary 4.8 in [19]; it has the property that every J-holomorphic map $\mathbb{R} \times S^1 \to \mathbb{R} \times M$ with $E(f) < E_0$ and $E_\omega(f) < \beta_0$ is a trivial cylinder over a closed Reeb orbit [19, Corollary 4.8].

The monotonicity lemma requires some care because the injectivity radius of Σ_k tends to zero. Following [19], we introduce J_k-invariant metrics g_k on

X_k which are $\mathbb{R}$-invariant on the necks $[-k-\varepsilon, \varepsilon] \times M$. We pick a positive $\varepsilon_1 < \min(\varepsilon, \delta, 1/2)$ smaller than the injectivity radii of (X_k, g_k) and $(\Sigma_k^\delta \times X_k, \widehat{g}_k)$, where $\widehat{g}_k = \sigma_k(\cdot, j_k \cdot) + g_k$ and Σ_k^δ denotes the δ-thick part of Σ_k. Let $\varepsilon_{\rm ML}$, $C_{\rm ML}$ be constants for which the monotonicity lemma holds in (X_k, J_k, g_k) and $(\Sigma_k^\delta \times X_k, \widehat{J}_k, \widehat{g}_k)$. Then with $\varepsilon_0 := \min(\varepsilon_1, \varepsilon_{\rm ML})$ the monotonicity lemma holds in the following form which replaces [19, Lemma 5.1]: For any K_k-holomorphic map $f : \Sigma \to X_k$ from a compact Riemann surface, passing through a point x with $f(\partial\Sigma)$ outside the g_k-ball $B_\rho(x)$ of radius $\rho < \varepsilon_0$, we have $\mathrm{area}_{\widehat{g}_k}(\widehat{f}) \geq C_{\rm ML}\rho^2$.

As in [19], we define for $-k+1 \leq R < S \leq -1$ with $S - R \geq 2$ subsets $\Sigma_R^S(f_k) \subset \Sigma_k$ by removing certain local minima and maxima[2] of the $\mathbb{R}$-projection of f_k from the preimages $f_k^{-1}([R, S] \times M) = \widehat{f}_k^{-1}([R, S] \times \widehat{M})$. For each k consider the functions

$$\chi_k : [-k+1, -3] \to \mathbb{Z}, \qquad r \mapsto \chi\left(\Sigma_r^{-1}(f_k)\right),$$
$$\widehat{\mathcal{A}}_k : [-k, 0] \to \mathbb{R}, \qquad r \mapsto \int_{f_k^{-1}([r,0]\times M)} \widehat{f}_k^* \widehat{\omega}_M.$$

Note that the function $\widehat{\mathcal{A}}_k$ is strictly decreasing because $\widehat{f}_k^*\widehat{\omega}_M \geq 0$. We define *essential levels* $r \in [-k, 0]$ for f_k as in [19]. Roughly speaking, a level is essential if either the topology of $\Sigma_r^{-1}(f_k)$ changes (in a sense made precise in [19]), or $\widehat{\mathcal{A}}_k(r)$ is an integer multiple of β_0. Using the version of the monotonicity lemma above, it follows as in [19] that the number of essential levels is bounded by a constant independent of k.

As in [19], a suitable selection of essential levels $-k = r_k^{(0)} < \cdots < r_k^{(N+1)} = 0$ gives rise to a decomposition of Σ_k into *essential regions* $\mathcal{E}_k^{(\nu)}$ and *cylindrical regions* $\mathcal{Z}_k^{(\nu)}$ defined by

$$\mathcal{E}_k^{(0)} := \Sigma_{-R_0}^{-k+R_0}(f_k), \qquad \mathcal{E}_k^{(\nu)} := \Sigma_{r_k^{(\nu)}-R_0}^{r_k^{(\nu)}+R_0}(f_k), \qquad \nu = 1, \ldots, N,$$
$$\mathcal{Z}_k^{(\nu)} := \Sigma_{r_k^{(\nu)}+R_0}^{r_k^{(\nu+1)}-R_0}(f_k), \qquad \nu = 0, \ldots, N$$

for a suitable constant $R_0 > 0$. Here N, R_0 are independent of k and the regions have the following properties:

(i) the topology of the surfaces $\mathcal{E}_k^{(\nu)}$ and $\mathcal{Z}_k^{(\nu)}$ is independent of k;
(ii) each component of $\mathcal{Z}_k^{(\nu)}$ is a cylinder whose modulus tends to infinity as $k \to \infty$.

[2] Local maxima actually do not occur in the contact case.

Let $\chi_1 \leq 0$ be the minimal value of the Euler characteristic of a component of an essential region $\mathcal{E}_k^{(\nu)}$. Set

$$\ell_1 := \ell - 2\pi\chi_1/\delta^2 \geq \ell$$

with the constant ℓ in [19, Lemma 4.10] (for ordinary J-holomorphic cylinders). Fix some $L \geq 1$.

For large k, the ith component of a cylindrical region $\mathcal{Z}_k^{(\nu)}$ is biholomorphic to $[-L_k^i - 5L - \ell_1, L_k^i + 5L + \ell_1] \times \mathbb{R}/\mathbb{Z}$ for some $L_k^i \geq 0$ tending to ∞ as $k \to \infty$. Due to the choice of ℓ_1 and Lemma D.3 above, the smaller cylinder $[-L_k^i - 5L - \ell, L_k^i + 5L + \ell] \times \mathbb{R}/\mathbb{Z}$ is contained in the δ-thin part of Σ_k. Since the inhomogeneous perturbation K_k vanishes on the δ-thin part, f_k restricts to an ordinary J_k-holomorphic map on $[-L_k^i - 5L - \ell, L_k^i + 5L + \ell] \times \mathbb{R}/\mathbb{Z}$. As in [19] now it follows that the restriction of f_k to $[-L_k^i - 5L, L_k^i + 5L] \times \mathbb{R}/\mathbb{Z}$ converges in C^∞_{loc} to a trivial cylinder over a closed Reeb orbit. This establishes convergence on the cylindrical regions.

Next consider an essential region $\mathcal{E}_k^{(\nu)}$. Define a larger region $E_k^{(\nu)}$ by gluing to each boundary component a cylinder of modulus $5L$ from the adjacent cylindrical component. Define the maps

$$f_k^{(0)} : E_k^{(0)} \to X_+ \amalg X_- \quad \text{and} \quad f_k^{(\nu)} : E_k^{(\nu)} \to \mathbb{R} \times M, \qquad \nu = 1, \dots, N$$

as the restriction of f_k, shifted by $-r_k^{(\nu)}$ in the $\mathbb{R}$-component for $\nu \geq 1$. Consider the sequence $f_k^{(\nu)}$ for fixed $\nu \geq 1$ (the case $\nu = 0$ differs only in notation). They are K_k-holomorphic and their images are contained in a compact region $[-R_1, R_1] \times M$ for all k [19, Lemma 5.12]. Moreover, their restrictions to the boundary collars of modulus $5L$ are J_k-holomorphic and converge to cylinders over Reeb orbits, so the satisfy condition **(A5)** with a uniform constant D. Hence they satisfy the hypotheses of Proposition D.5 above and we obtain convergence on the essential regions to nodal K-holomorphic maps.

The construction of the limit domain (Σ^*, j) from the essential and cylindrical regions and the proof of convergence to a broken K-holomorphic map $F : (\Sigma^*, j) \to (X^*, J^*, K)$ work exactly as in [19] now. This concludes the proof of Theorem D.1. $\square$

E Jet transversality

In this appendix we prove a jet transversality result which is needed in the proof of Lemma 6.6. The proof that follows is due to C. Wendl. Recall that a subset of a topological space is called *Baire subset* if it contains a countable intersection of open dense sets.

Proposition E.1 *Assume B is a smooth manifold equipped with a smooth m-dimensional foliation $\mathcal{F}$, $E \to B$ is a smooth real vector bundle of rank q, $B_0 \subset B$ is an open subset with compact closure, $S_{\text{fix}} \in \Gamma(E)$ is a smooth section, and*

$$\Gamma(E\,;\, B_0, S_{\text{fix}}) \subset \Gamma(E)$$

denotes the space of all smooth sections $S : B \to E$ such that $S \equiv S_{\text{fix}}$ on $B \backslash B_0$. Then for every integer $k > m - q$, there exists a Baire (in the C^∞-topology) subset $\Gamma^*(E\,;\, B_0, S_{\text{fix}}) \subset \Gamma(E\,;\, B_0, S_{\text{fix}})$ such that for each $S \in \Gamma^*(E\,;\, B_0, S_{\text{fix}})$, there is no embedded k-disk in the intersection of $S^{-1}(0) \cap B_0$ with any leaf of $\mathcal{F}$.

The same result holds true within the subspace $\Gamma^\varepsilon(E\,;\, B_0, S_{\text{fix}}) \subset \Gamma(E\,;\, B_0, S_{\text{fix}})$ of those sections whose difference to S_{fix} belongs to the class of Floer's C^ε-sections (see Sect. 7), equipped with the C^ε-norm on the difference.

The intuition behind this statement is as follows. For $b \in B_0$, let $\mathcal{F}_b$ denote the leaf of $\mathcal{F}$ through b. If one could arrange for every $b \in B_0$ that $S|_{\mathcal{F}_b} : \mathcal{F}_b \to E$ is transverse to the zero-section, then the result would be immediate because $S^{-1}(0) \cap \mathcal{F}_b$ would be a smooth submanifold with dimension $m - q$, which is less than k. Of course this transversality condition can be arranged for generic S for any individual leaf, but not for all leaves simultaneously—nonetheless, the existence of a k-disk in $S^{-1}(0) \cap \mathcal{F}_b$ for some $b \in B_0$ and $k > m - q$ represents an extremely strong failure of transversality, which can be measured quantitatively by looking at arbitrarily high-order jets of S in directions tangent to leaves. We will show that for each $r \geq 1$ and for generic $S \in \Gamma(E\,;\, B_0, S_{\text{fix}})$, applying the appropriate r-jet condition to $S^{-1}(0)$ reduces its dimension by an amount that becomes negative as soon as r is sufficiently large, hence $S^{-1}(0)$ contains no points in B_0 at which the r-jet condition is satisfied for every r. The main technical tool needed for this argument is the Sard-Smale theorem.

Proof Note that the result is trivially true if $k > m$, so we will assume from now on that $k \leq m$. Let $n = \dim B$. Recall that for $b \in B$, two smooth sections of E defined near b are said to be *r-tangent at b* if their values and derivatives of all orders up to r (computed in any choice of local coordinates and trivialization) match. The r-tangency classes of local sections at b are called *r-jets at b*, and the union for all $b \in B$ of the spaces of r-jets at b forms a smooth fibre bundle $E^{(r)} \to B$ whose fibres have dimension $q\frac{(n+r)!}{n!r!}$, see e.g. [29, §1.2].

Here is a related notion. If $\mathcal{F}_b$ denotes the leaf of $\mathcal{F}$ through b, we will say that two smooth k-dimensional submanifolds $Y, Y' \subset \mathcal{F}_b$ containing b are *r-tangent at b* if there exists a smooth coordinate chart identifying b with $0 \in \mathbb{R}^n$ such that a neighbourhood of b in $\mathcal{F}_b$ looks like $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$ and neighbourhoods of b in both Y and Y' look like graphs

$$\{(x, f(x), 0) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \times \mathbb{R}^{n-m}\}$$

of functions $f : \mathbb{R}^k \to \mathbb{R}^{m-k}$ that vanish up to order r at 0. This notion of tangency defines an equivalence relation for k-dimensional submanifolds $Y \subset \mathcal{F}_b$ through b; we shall denote by $[Y]_b$ the equivalence class represented by Y and denote the set of all such r-tangency classes at b by $\Upsilon_b^{r,k}$. Since $\Upsilon_b^{r,k}$ can locally be parametrized by the space of r-jets of maps $f : \mathbb{R}^k \to \mathbb{R}^{m-k}$ with $f(0) = 0$, it is a smooth manifold of dimension $(m-k)\left(\frac{(k+r)!}{k!r!} - 1\right)$, and $\Upsilon^{r,k} := \bigcup_{b \in B} \Upsilon_b^{r,k}$ is a smooth fibre bundle over B.

For $S \in \Gamma(E)$ and each choice of integers $r, k \geq 0$, define

$$\mathcal{Z}^{r,k}(S) \subset \Upsilon^{r,k}$$

to be the set of all r-tangency classes $[Y]_b \in \Upsilon^{r,k}$ at points $b \in B_0$ such that S is r-tangent at b to a section that vanishes along some k-dimensional submanifold of $\mathcal{F}_b$ representing $[Y]_b$. Notice that $\mathcal{Z}^{r,k}(S)$ is just $S^{-1}(0) \cap B_0$ whenever either k or r vanishes. We will be most interested in cases where $m - q < k \leq m$ and $r > 0$ is large.

Near any point $[Y_0]_{b_0} \in \mathcal{Z}^{r,k}(S)$, one can describe the local structure of $\mathcal{Z}^{r,k}(S)$ as follows. Choose neighbourhoods $\mathcal{U} \subset B_0$ of b_0 and $\mathcal{V} \subset \Upsilon^{r,k}$ of $[Y_0]_{b_0}$, and a smooth family of charts $\varphi_{[Y]_b} : \mathcal{U} \to \mathbb{R}^n$ parametrized by elements $[Y]_b \in \mathcal{V}$ such that $\varphi_{[Y]_b}(b) = 0$ and the r-tangency class of $\varphi_{[Y]_b}^{-1}(\mathbb{R}^k \times \{0\})$ at b is $[Y]_b$. Fix also a local trivialization of $E \to B$ over $\mathcal{U}$. Now for any $[Y]_b \in \mathcal{V}$, the local trivialization together with the chart $\varphi_{[Y]_b}$ identifies S near b with a smooth $\mathbb{R}^q$-valued function $f_{[Y]_b}$ defined on a neighbourhood of the origin in $\mathbb{R}^n$, and we have $[Y]_b \in \mathcal{Z}^{r,k}(S)$ if and only if the restriction of $f_{[Y]_b}$ to $\mathbb{R}^k \times \{0\}$ vanishes up to order r at 0, i.e. if all terms in its Taylor polynomial of degree r involving only the first k variables are zero. This defines a smooth map

$$\Phi_S : \mathcal{V} \to \mathbb{R}^{q(k+r)!/(k!r!)}$$

whose zero set is a neighbourhood of $[Y_0]_{b_0}$ in $\mathcal{Z}^{r,k}(S)$. Let us call the element $[Y_0]_{b_0} \in \mathcal{Z}^{r,k}(S)$ *regular* if it is a regular point of this map; one can check that this notion does not depend on the choices involved. If all elements in $\mathcal{Z}^{r,k}(S)$ are regular, then it is a smooth manifold with

$$\begin{aligned}\dim \mathcal{Z}^{r,k}(S) &= \dim \Upsilon^{r,k} - q\frac{(k+r)!}{k!r!} = n + (m-k)\left(\frac{(k+r)!}{k!r!} - 1\right) - q\frac{(k+r)!}{k!r!} \\ &= n - q - [k - (m-q)]\left(\frac{(k+r)!}{k!r!} - 1\right).\end{aligned} \tag{26}$$

Now define

$$\Gamma^*(E\,;\,B_0, S_{\text{fix}}) \subset \Gamma(E\,;\,B_0, S_{\text{fix}})$$

as the subset consisting of all $S \in \Gamma(E\,;\,B_0, S_{\text{fix}})$ for which every $[Y]_b \in \mathcal{Z}^{r,k}(S)$ is regular. We claim that $\Gamma^*(E\,;\,B_0, S_{\text{fix}})$ is a Baire subset of $\Gamma(E\,;\,B_0, S_{\text{fix}})$. The first step is to prove that it is dense. To this end, fix an arbitrary element $S_{\text{ref}} \subset \Gamma(E\,;\,B_0, S_{\text{fix}})$, choose an appropriate sequence of decaying positive numbers $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}}$, and let

$$\Gamma^\varepsilon \subset \Gamma(E\,;\,B_0, S_{\text{fix}})$$

denote the space of all smooth sections $S : B \to E$ such that $S - S_{\text{ref}}$ belongs to the space of Floer's C^ε-sections $B \to E$ that vanish outside B_0. This makes Γ^ε a separable Banach manifold covered by a single chart. We then define the topological space

$$\mathcal{Z}^{r,k}(\Gamma^\varepsilon) = \left\{(S, [Y]_b) \in \Gamma^\epsilon \times \Upsilon^{r,k} \;\middle|\; [Y]_b \in \mathcal{Z}^{r,k}(S)\right\}.$$

A neighbourhood in $\mathcal{Z}^{r,k}(\Gamma^\varepsilon)$ of any given element $(S_0, [Y_0]_{b_0})$ can be described in terms of the map Φ_S defined above as the zero set of the smooth map

$$\Phi : \Gamma^\varepsilon \times \mathcal{V} \to \mathbb{R}^{q(k+r)!/(k!r!)} : (S, [Y]_b) \mapsto \Phi_S([Y]_b).$$

Its partial derivative at $(S_0, [Y_0]_{b_0})$ with respect to the first factor gives a linear map that associates to a C^ε-smooth section $S \in T_{S_0}\Gamma^\varepsilon$ the first k components of its r-jet at b_0 (in the local coordinates above). Since $b_0 \in B_0$, one can choose S to realize any desired r-jet at b_0 by setting it equal to the appropriate polynomial near b_0 and then multiplying this by a C^ε-smooth bump function supported in B_0. It follows that the derivative of Φ is surjective on its zero set, and $\mathcal{Z}^{r,k}(\Gamma^\varepsilon)$ is therefore a smooth Banach submanifold of $\Gamma^\epsilon \times \Upsilon^{r,k}$. Applying the Sard-Smale theorem to the natural projection $\mathcal{Z}^{r,k}(\Gamma^\varepsilon) \to \Gamma^\varepsilon$, $(S, [Y]_b) \mapsto S$ then provides a Baire subset $\Gamma^{\varepsilon,*}$ of Γ^ε consisting of sections S for which all elements of $\mathcal{Z}^{r,k}(S)$ are regular. In particular, since S_{ref} was arbitrary and the C^ε-space has a continuous inclusion into C^∞, this implies that $\Gamma^*(E\,;\,B_0, S_{\text{fix}})$ is dense in $\Gamma(E\,;\,B_0, S_{\text{fix}})$ in the C^∞-topology.

Finally, choose an exhaustion of $\Upsilon^{r,k}|_{B_0}$ by a countable sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots$, and for each $N \in \mathbb{N}$, define $\Gamma^*_N(E\,;\,B_0, S_{\text{fix}}) \subset \Gamma(E\,;\,B_0, S_{\text{fix}})$ as the subset consisting of all $S \in \Gamma(E\,;\,B_0, S_{\text{fix}})$ for which every point $[Y]_b \in \mathcal{Z}^{r,k}(S) \cap K_N$ is regular. The previous argument shows that each of these sets is dense in $\Gamma(E\,;\,B_0, S_{\text{fix}})$, but they are also open due to the compactness of K_N. The intersection of all of them is $\Gamma^*(E\,;\,B_0, S_{\mathrm{fix}})$, which is therefore a Baire subset.

Now to conclude, assume that $m - q < k$. The dimension formula (26) then shows that we can make $\dim \mathcal{Z}^{r,k}(S) < 0$ by choosing $r \in \mathbb{N}$ sufficiently large. Having done this, $\mathcal{Z}^{r,k}(S)$ is empty for every $S \in \Gamma^*(E\,;\,B_0, S_{\mathrm{fix}})$, but if there were a k-dimensional submanifold $Y \subset \mathcal{F}_b$ through some point $b \in B_0$ with S vanishing on Y, it would represent an element $[Y]_b \in \mathcal{Z}^{r,k}(S)$, giving a contradiction.

This concludes the proof for the space $\Gamma(E\,;\,B_0, S_{\mathrm{fix}})$. Within the subspace $\Gamma^\varepsilon(E\,;\,B_0, S_{\mathrm{fix}}) \subset \Gamma(E\,;\,B_0, S_{\mathrm{fix}})$ the proof is identical up to the construction of the Baire subset $\Gamma^{\varepsilon,*}$, noting that $\Gamma^\varepsilon = \Gamma^\varepsilon(E\,;\,B_0, S_{\mathrm{fix}})$ because $S_{\mathrm{ref}} \in \Gamma^\varepsilon(E\,;\,B_0, S_{\mathrm{fix}})$. Now $\Gamma^{\varepsilon,*}$ is already the desired Baire subset, so the exhaustion argument is not necessary and the proof is concluded as before. □

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